Reviews in Mathematical Physics
Vol. 23, No. 2 (2011) 211–232

DOI: 10.1142/S0129055X11004291

RUELLE–LANFORD FUNCTIONS AND LARGE DEVIATIONS FOR ASYMPTOTICALLY DECOUPLED QUANTUM SYSTEMS

YOSHIKO OGATA

Graduate School of Mathematics,
University of Tokyo, Japan
yoshiko@ms.u-tokyo.ac.jp

LUC REY-BELLET

Department of Mathematics and Statistics,
University of Massachusetts Amherst, USA
luc@math.umass.edu

Received 22 September 2010
Revised 2 February 2011

We recover, expand, and unify quantum (and classical) large deviation results for lattice Gibbs states. The main new ingredient in this paper is a control on the overlap of spectral projections for non-commutative observables. Our proof of large deviations is based on Ruelle–Lanford functions [20, 34] which establishes the existence of a rate function directly by subadditivity arguments, as done in the classical case in [23, 32], instead of relying on Gärtner–Ellis theorem, and cluster expansion or transfer operators as done in the quantum case in [21, 13, 27, 22, 16, 28]. We assume that the Gibbs states are asymptotically decoupled [23, 32], which controls the dependence of observables localized at different spatial locations. In the companion paper [29], we discuss the characterization of rate functions in terms of relative entropies.

Keywords: Quantum spin systems; large deviations; thermodynamics formalism; quantum statistical mechanics.

Mathematics Subject Classification 2010: 82B10, 60F10

1. Introduction

Consider a spin system on the lattice $\mathbb{Z}^d$ in thermal equilibrium described by a state ω. For a region $\Lambda \subset \mathbb{Z}^d$, let K_Λ be a macroscopic observable, for example the total energy or the total magnetization in the region Λ. One expects, as a rule, that such observables have a distribution which is very sharply concentrated around the equilibrium mean value and that the fluctuations of such observables are exponentially small in the volume $|\Lambda|$ of the domain, except at a first order phase transition where coexisting phases can induce macroscopically large fluctuations. This property is expressed by a *large deviation principle*: For a Borel set $A \subset \mathbb{R}$

let $\mathbf{I}_A(K_\Lambda/|\Lambda|)$ denote the spectral projection onto the eigenspace of $K_\Lambda/|\Lambda|$ corresponding to eigenvalues of $K_\Lambda/|\Lambda|$ in the set A. A large deviation principle holds if there exists a rate function $s(x)$ such that

$$\omega(\mathbf{I}_A(K_\Lambda/|\Lambda|)) \asymp \exp\left(|\Lambda| \sup_{x\in A} s(x)\right).$$

In classical mechanics systems this problem is mathematically very well-understood and very general *large deviations theorems* have been proved both for systems on a lattice or in the continuum see [20, 34, 30, 12, 6, 10, 14, 15, 23, 32, 33]. For quantum mechanical systems, the problem of large deviations has, in comparison, received little attention and is only partially understood. The difficulty lies, partly, in the non-commutativity of quantum mechanical observables but also at a deeper level, in the lack of control on the boundary effects in quantum mechanics. Known bulk/boundary estimates are sufficient to prove the existence of thermodynamic functions such as entropy and free energy, see e.g., [35, 19, 5, 36] but they are, so far, *not* sufficient to prove general large deviation results, especially at low temperatures for spatial dimension more than one. A number of quantum large deviation results have been proved in the past few years [31, 21, 13, 22, 27, 16, 17, 7, 28, 8], (see also [4] for an information-theoretic interpretation of relative entropy). Common to all these papers is that the large deviation results are obtained by an application of Gärtner–Ellis theorem, in particular the smoothness of the logarithmic moment generating functions (i.e. a suitable free energy functional) is necessary and is proved by cluster expansion or using a transfer operator.

For classical Gibbs states there are several different proofs of the large deviation principle and we follow here the approach by Lewis, Pfister and Sullivan [23, 32]. In this approach, the state ω of the infinite system is assumed to be *asymptotically decoupled* (see Sec. 3.2 for a formal and more general definition): Given a finite region $V \subset \mathbb{Z}^d$ there exists a function $c(V)$ such that if A is a nonnegative observable supported in the region V and B is a nonnegative observable supported in the complement $\mathbb{Z}^d \backslash V$ one has

$$e^{-c(V)}\omega(A)\omega(B) \leq \omega(AB) \leq \omega(A)\omega(B)e^{c(V)},$$

with

$$\lim_{V\nearrow\mathbb{Z}^d} \frac{c(V)}{|V|} = 0.$$

In the classical case, this property is a fairly easy consequence of the DLR equation for Gibbs states. Using this property and subadditivity arguments one proves then directly the existence of a rate function $s(x)$ for the observable of interest. This general approach to large deviation (summarized in Sec. 2) was pioneered by Lanford and Ruelle papers [20, 34] and we follow here the terminology of [23, 32].

To use this strategy for quantum systems we face two obstacles. The first one is that it is not known whether a Gibbs-KMS state for a quantum spin system is asymptotically decoupled in general. This property is only known to hold in spatial dimension one (proved by Araki in [1]) and at high temperatures in arbitrary dimension (see, e.g., [2]). We believe that it is an important open problem to determine whether this property holds for general quantum spin systems or not but there is no new result in this direction in this paper. The second obstacle lies in the generalization of the subadditivity argument of [23, 32] to general non-commutative observables. The new ingredient needed here is a control on the overlap of spectral projections for K_Λ and the spectral projections for $K_{\Lambda_1} + K_{\Lambda_2}$ with $\Lambda = \Lambda_1 \cup \Lambda_2$, which differs from K_Λ by a boundary term. This new estimate is proved in Sec. 4.4, see Proposition 4.10.

Using this approach we are able to recover, unify, and extend slightly the known large deviation results for quantum (and classical) spin systems. In addition the proofs given here are quite short and self-contained.

This paper is organized as follows. In Sec. 2, we give a brief exposition of the road to large deviation via proving the existence of the Ruelle–Lanford function which is an Boltzmann entropy-like functional. In Sec. 3, we recall the elements of the quantum spin system formalism needed in the paper and we introduce the asymptotic decoupling condition for states of quantum systems which is central in our analysis. In Sec. 4, we prove large deviation theorems for three different cases: (a) Commuting observables, (b) Classical observables, (c) General finite-range observables in dimension 1. The discussion of the rate functions and their characterization in terms of relative entropies is in the companion paper [29].

2. Ruelle–Lanford Functions

Let X be a complete metric space, let $\{\mu_n\}$ be a sequence of Borel probability measures on X, and let $\{v_n\}$ an increasing sequence of positive numbers with $\lim_{n\to\infty} v_n = +\infty$. We say that μ_n satisfies a *large deviation principle* (*LDP*) on the scale v_n if there exists a function $I : X \to [0,\infty]$, lower semicontinuous and with compact level sets, such that for any closed set C

$$\limsup_{n\to\infty} \frac{1}{v_n} \log \mu_n(C) \le -\inf_{x\in C} I(x), \tag{2.1}$$

and for any open set O

$$-\inf_{x\in O} I(x) \le \liminf_{n\to\infty} \frac{1}{v_n} \log \mu_n(O). \tag{2.2}$$

The function I is called the *rate function* for the LDP.

In statistical mechanics applications the measures μ_n are often distributions of sums of $\mathbb{R}$- or $\mathbb{R}^d$- valued weakly dependent random variables. One standard approach to prove an LDP is to combine the exponential Markov inequality for the upper bound (2.1) and a change of measure and ergodicity argument for the lower bound (2.2) (see, e.g., the proofs of Cramer and Gärtner–Ellis theorem in [9]). In the presence of phase transitions, i.e. lack of ergodicity with respect to spatial translation, additional arguments are needed to provide a lower bound. For example, in [12], the lower bound for the LDP for classical lattice Gibbs states is obtained by using the Shannon–McMillan theorem and an approximation argument by ergodic states.

Another route to LDP's using subadditivity arguments, much in the spirit of statistical mechanics, was pioneered in a remarkable paper by Lanford [20], itself based on earlier work by Ruelle [35]. We follow closely here the presentation in [23], see also [32].

For Borel sets B let us define the set functions

$$\overline{m}(B) = \limsup_{n\to\infty} \frac{1}{v_n} \log \mu_n(B), \quad \underline{m}(B) = \liminf_{n\to\infty} \frac{1}{v_n} \log \mu_n(B). \tag{2.3}$$

One has the elementary properties

(1) For any Borel set B, we have $-\infty \le \underline{m}(B) \le \overline{m}(B) \le 0$.
(2) If $B_1 \subset B_2$, then $\underline{m}(B_1) \le \underline{m}(B_2)$ and $\overline{m}(B_1) \le \overline{m}(B_2)$.
(3) For all B_1, B_2, we have $\overline{m}(B_1 \cup B_2) = \max\{\overline{m}(B_1), \overline{m}(B_2)\}$.

Property (3) is an key property in large deviations and is usually refereed to as the *principle of the largest term*: large deviations occur in the least unlikely way of all possible ways.

Let $B_\varepsilon(x)$ denote the ball of radius ε centered at x and let us define

$$\overline{s}(x) = \inf_\varepsilon \overline{m}(B_\varepsilon(x)), \quad \underline{s}(x) = \inf_\varepsilon \underline{m}(B_\varepsilon(x)). \tag{2.4}$$

Definition 2.1. The pair (μ_n, v_n) has a *Ruelle–Lanford function* (RL-function) $s(x)$ if

$$\overline{s}(x) = \underline{s}(x),$$

for all $x \in X$. In this case we set $s(x) = \overline{s}(x) = \underline{s}(x)$.

The next proposition is standard and shows that the existence of RL-function (almost) implies the existence of a LDP.

Proposition 2.2. *The Ruelle–Lanford function $s(x)$ is upper semicontinuous and*

$$\underline{m}(O) \ge \sup_{x\in O} s(x), \quad O \textit{ open}, \tag{2.5}$$

$$\overline{m}(K) \le \sup_{x\in K} s(x), \quad K \textit{ compact}. \tag{2.6}$$

Proof (Sketch). The upper semicontinuity follows from the definition. The lower bound is immediate: For any $x \in O$ and ε sufficiently small we have $\underline{m}(O) \ge \underline{m}(B_\varepsilon(x))$ and thus $\underline{m}(O) \ge \underline{s}(x) = s(x)$ for all $x \in O$.

To prove the upper bound, given $\varepsilon > 0$ we cover the compact set K by $N = N(\varepsilon)$ balls $B_\varepsilon(x_l)$ with centers in $x_l \in K$. Using Properties (2) and (3) we have

$$\overline{m}(K) \le \overline{m}\left(\bigcup_{l=1}^N B_\varepsilon(x_l)\right) \le \max_l \overline{m}(B_\varepsilon(x_l)) \le \sup_{x\in K} \overline{m}(B_\varepsilon(x)).$$

Since ε is arbitrary the upper bound follows. □

The statement in Proposition 2.2 is usually referred to as a *weak large deviation principle* since the upper bound holds only for compact sets. In the problems discussed in this paper, the probability measures μ_n are supported uniformly on compact sets and the previous lemma yields immediately a large deviation principle with rate function $-s(x)$. More generally one obtains a large deviation principle by combining Proposition 2.2 with a proof that the sequence of probability measures μ_n is *exponentially tight* (see, e.g., [9, Sec. 1.2]).

To identify the rate function we use a standard large deviation result.

Proposition 2.3 (Laplace–Varadhan's Lemma). *Suppose that μ_n satisfies a large deviation principle on the scale v_n with rate function $I(x)$. Let f be any continuous function and suppose that for some $\gamma > 1$ we have the moment condition $\limsup_{n\to\infty} \frac{1}{v_n} \log \mu_n(e^{\gamma v_n f(x)}) < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{v_n} \log \mu_n(e^{v_n f(x)}) = \sup_x (f(x) - I(x)).$$

If $X = \mathbb{R}^n$ and $f(x) = \alpha \cdot x$, we obtain

$$e(\alpha) \equiv \lim_{n\to\infty} \frac{1}{v_n} \log \mu_n(e^{v_n \alpha\cdot x}) = \sup_x (\alpha \cdot x + s(x)),$$

i.e. the moment generating function of μ_n is the Legendre transform of $-s(x)$. If, in addition, *we know, a priori*, that the rate function $s(x)$ is *concave* then by convex duality we obtain that

$$s(x) = \inf_\alpha (e(\alpha) - \alpha \cdot x),$$

that is, the rate function is the Legendre transform of the logarithmic moment generating function. Note that in our examples the moment condition will be trivially satisfied.

3. Quantum Lattice Systems

3.1. *Interactions and states*

We introduce some notations and briefly recall the mathematical framework for quantum spin systems, [19, 36, 5, 3].

***C^**-algebras.** Let $\mathcal{A}$ be a finite-dimensional C^*-algebra. For any finite subset $\Lambda \subset \mathbb{Z}^d$, let $\mathcal{O}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{O}_x$ where $\mathcal{O}_x$ is isomorphic to $\mathcal{A}$. If $\Lambda \subset \Lambda'$, there is a natural embedding $\mathcal{O}_\Lambda$ into $\mathcal{O}_{\Lambda'}$ and the algebras $\{\mathcal{O}_\Lambda\}_{\Lambda\subset\mathbb{Z}^d,\text{finite}}$ form a partially ordered family of matrix algebras. The *algebra of observables* for the infinite system is given by the C^*-inductive limit $\mathcal{O}$ of $\bigcup_{\Lambda\subset\mathbb{Z}^d,\text{finite}} \mathcal{O}_\Lambda$.

States. Let ω be a state on $\mathcal{O}$, i.e. ω is a positive, normalized linear functional on $\mathcal{O}$. Let $\{\tau_x\}_{x\in\mathbb{Z}^d}$ denote the group of spatial translations. A state ω is called *translation invariant* if $\omega(\tau_x A) = \omega(A)$ for all $x \in \mathbb{Z}^d$ and all $A \in \mathcal{O}$. The action of $\mathbb{Z}^d$ on $\mathcal{O}$ is asymptotically abelian [5] and thus the set of translation invariant states is a simplex. We say that a state is *ergodic* if it is an extremal point of this simplex.

Classical subalgebras and states. A standard probabilistic setting is recovered by considering commutative (sub)algebras. Let $\mathcal{A}^{(cl)}$ be an abelian subalgebra of $\mathcal{A}$ with $N = \dim \mathcal{A}^{(cl)}$. For finite subsets Λ of $\mathbb{Z}^d$ let $\mathcal{O}_\Lambda^{(cl)} = \bigotimes_{x\in\Lambda} \mathcal{O}_x^{(cl)}$ with $\mathcal{O}_x^{(cl)}$ is isomorphic to $\mathcal{A}^{(cl)}$. We denote by $\mathcal{O}^{(cl)}$ the inductive limit of $\bigcup_{\Lambda\subset\mathbb{Z}^d,\text{finite}} \mathcal{O}_\Lambda^{(cl)}$. The commutative algebra $\mathcal{O}^{(cl)}$ can be identified with $C(\mathcal{L})$ where $\mathcal{L} = \{1, \dots, N\}^{\mathbb{Z}^d}$ with product topology is called a *classical* C^*-algebra. The restriction of any state ω on $\mathcal{O}$ gives a normalized linear functional $\omega^{(cl)}$ on $\mathcal{O}^{(cl)}$. By Riesz Markov Theorem there exists a probability measure $d\omega^{(cl)}$ such that for any $A \in \mathcal{O}^{(cl)}$

$$\omega(A) = \omega^{(cl)}(A) = \int_{\mathcal{L}} A(l) d\omega(l).$$

Interactions and Hamiltonians. An *interaction* $\Psi = \{\psi_X\}_{X\subset\mathbb{Z}^d,\text{finite}}$ is a map from the the finite subsets of $\mathbb{Z}^d$ to selfadjoint elements ψ_X in $\mathcal{O}_X$. We will assume throughout this paper that Ψ is translation invariant, i.e. $\tau_x(\psi_X) = \psi_{X+x}$ for any $X \subset \mathbb{Z}^d$ and any $x \in \mathbb{Z}^d$. An interaction Ψ is *classical* if there exists a classical C^*-subalgebra $\mathcal{O}^{(cl)}$ such that $\psi_X \in \mathcal{O}^{(cl)}$ for all $X \subset \mathbb{Z}^d$.

We equip translation invariant interactions Ψ with the norm

$$\|\Psi\| \equiv \sum_{X\ni 0} |X|^{-1} \|\psi_X\|,$$

where $|X|$ is the cardinality of the set X and denote by $\mathcal{B}$ the corresponding Banach space. To any interaction $\Psi \in \mathcal{B}$ we associate *Hamiltonians* (or *macroscopic observables*) $K_\Lambda = K_\Lambda(\Psi)$: For $\Lambda \subset \mathbb{Z}^d$ finite we define

$$K_\Lambda = \sum_{X\subset\Lambda} \psi_X.$$

Furthermore, to any $\Psi \in \mathcal{B}$, we associate an observable in $\mathcal{O}$ by

$$A_\Psi = \sum_{X\ni 0} \frac{1}{|X|} \psi_X.$$

When we consider Gibbs state, two kinds of interactions Ψ and Φ will be considered. The interaction Ψ corresponds to the observables while Φ defines the Gibbs state. We denote by K_Λ the local Hamiltonian associated with Ψ and by H_Λ associated with Φ.

Large deviations. For $n \in \mathbb{N}$ let $\Lambda(n) = \{z \in \mathbb{Z}^d; 0 \le z_i \le n-1\}$ denote the cube with $|\Lambda(n)| = n^d$ lattice points and left hand corner at the origin. If ω is an ergodic state then the von Neumann ergodic theorem implies that

$$\lim_{n\to\infty} \frac{1}{|\Lambda(n)|} K_{\Lambda(n)} = \omega(A_\Psi)$$

strongly in the GNS representation and it is natural to investigate the large deviation properties, on the scale $v_n = |\Lambda(n)|$, of the sequence of Borel measures on $\mathbb{R}$

$$\mu_n(A) \equiv \omega(\mathbf{I}_A(|\Lambda(n)|^{-1} K_{\Lambda(n)})),$$

where A is a Borel set and $\mathbf{I}_A(H)$ denotes the spectral projection onto the eigenspace of H spanned by the eigenvalues contained in the set A. We interpret the $\mu_n(A)$ as the probability that the observables $|\Lambda(n)|^{-1}K_{\Lambda(n)}$ takes value in A if the system is in the state ω.

3.2. *Asymptotically decoupled states*

The states we consider in this paper obey a property of weak dependence between disjoint regions of the lattice. We follow here the terminology used in [32] for the classical case.

Let $C(m)$ be an arbitrary cube of side length m and let us denote by $C^r(m)$ the cube of side length $m + 2r$ centered at the same point of $\mathbb{Z}^d$ as $C(m)$.

Definition 3.1. A state ω on $\mathcal{O}$ is *asymptotically decoupled* with parameters g and c if

(1) There exist a function $g : \mathbb{N} \to \mathbb{N}$ with $\lim_{m\to\infty} g(m)/m = 0$ and a function $c : \mathbb{N} \to [0, \infty)$ with $\lim_{m\to\infty} c(m)/|C(m)| = 0$.
(2) For any cube $C(m)$, $m \in \mathbb{N}$, any nonnegative $A \in \mathcal{O}_{C(m)}$, any nonnegative $B \in \mathcal{O}_{C^{g(m)}(m)^c}$ we have

$$e^{-c(m)}\omega(A)\omega(B) \le \omega(AB) \le e^{c(m)}\omega(A)\omega(B).$$

Examples of asymptotically decoupled states are

(a) **Product states.** Any product state ω_0 is asymptotically decoupled with parameters $c = g = 0$.
(b) **Classical Gibbs states.** Let $\mathcal{O}^{(cl)}$ be a classical C^*-algebra and let Φ be a classical translation invariant interaction such that $\|\Phi\|_0 \equiv \sum_{X\ni 0} \|\phi_x\|$ is finite. A Gibbs state for the interaction Φ is a probability measure $\omega^{(\Phi)}$ which satisfies the DLR equation (see, e.g., [35, 36]). Using the DLR equation one proves easily (see, e.g., [23, Sec. 9]) that for any positive $A \in \mathcal{O}_{C(m)}$ we have

$$e^{-c(m)}\omega^{(\Phi)}(A) \le \frac{\mathrm{tr}(Ae^{-H_\Lambda})}{\mathrm{tr}(e^{-H_\Lambda})} \le e^{c(m)}\omega^{(\Phi)}(A), \tag{3.1}$$

with $c(m) = \|W_{C(m)}\|$ where $W_{C(m)}$ is the boundary interaction $W_{C(m)} = \sum_{\substack{X\cap C(m)\neq\emptyset \\ X\cap C(m)^c\neq\emptyset}} \phi_X$. This implies easily that $\omega^{(\Phi)}$ is asymptotically decoupled if $\|\Phi\|_0 < \infty$.
(c) **Quantum KMS states.** Let Φ be a translation invariant interaction. A KMS state for the interaction Φ is a state which satisfies the KMS condition or equivalently the Gibbs condition which is a quantum analog of the DLR equation (see, e.g., [5, 36, 3] for an up-to-date presentation). It is *not known* if KMS-Gibbs states are asymptotically decoupled, in general. Let us assume however that [1, 2] either

(i) $d = 1$ and Φ finite range (i.e. for some $R > 0$ diamX $> R$ implies $\phi_X = 0$), or

(ii) d arbitrary and $\|\Phi\|_\lambda \equiv \sum_{X \ni 0} e^{\lambda|X|} \|\phi_X\|$ is sufficiently small,

then one can show that for a Gibbs-KMS state $\omega^{(\Phi)}$ and $A \in \mathcal{O}_{C(m)}$ we have the bound (3.1) where $c(m) = C(\Phi) \sum_{\substack{X \cap C(m) \neq \emptyset \\ X \cap C(m)^c \neq \emptyset}} \|\phi_X\|$. Contrary to the classical case the bound is highly nontrivial to prove and relies on the Gibbs condition, Araki perturbation theory, and control of imaginary-time dynamics. This bound implies that $\omega^{(\Phi)}$ is asymptotically decoupled.

(d) **Markov measures.** Let ω be a stationary Markov chain on a finite state space with transition matrix Q and invariant probability q. Then ω is asymptotically decoupled if and only if Q is irreducible and aperiodic (i.e. mixing). If m is the smallest integer such that Q^m has strictly positive entries then the parameters are

$$g(m) = m - 1, \quad c(n) = \sup_{\sigma_1, \sigma_2} \left| \log \frac{Q^m(\sigma_1, \sigma_2)}{q(\sigma_2)} \right|.$$

(e) **Finitely correlated states.** These states are a non-commutative generalization of Markov measures and are asymptotically decoupled if and only if they are mixing which occur under suitable conditions similar to the aperiodicity condition for Markov measures. See [18, 11, 28] for details.

4. Quantum Large Deviations Theorems

We prove several large deviations theorems for quantum states (in order of increasing difficulty) by showing the existence of concave RL-functions. This unifies, simplifies and extend a number of quantum large deviation results which have been proved with different techniques (Gärtner–Ellis Theorem via transfer operators, cluster expansions, etc.). Our proof have the advantage of being fairly short, self-contained, to apply in some situations where the rate function is not smooth.

4.1. *Preliminaries*

In this section we prove an energy estimate used throughout the paper and explain the strategy (after [23]) used to prove the existence of a concave Ruelle–Lanford function.

The first fact is a very slight variation on standard bulk/boundary energy estimate, see, e.g., [36, 5, 32]. Given integers n and m and a function $g(m)$ such that $\lim_{m \to \infty} g(m)/m = 0$ we choose k to be largest *even* integer such that

$$n = k(m + 2g(m)) + r, \quad 0 \leq r < 2(m + 2g(m)),$$

(having k even will be convenient in the sequel). We next decompose the cube $\Lambda(k(m + 2g(m))$ into k^d pairwise disjoint and contiguous cubes $\tilde{C}_j$, each of which are each translates of $\Lambda(m + 2g(m))$ and then further divide each cube $\tilde{C}_j$ into a cube C_j which is centered at the same point as $\tilde{C}_j$ and is a translate of $\Lambda(m)$ and a "corridor" $\tilde{C}_j \setminus C_j$ of width $g(m)$. We shall need estimates on the difference between the Hamiltonian $K_\Lambda(n)$ and the "decoupled" Hamiltonian for the collection of cubes C_j, i.e. $\sum_{j=1}^{k^d} K_{C_j}$.

Lemma 4.1. *Let Ψ be an interaction with $\|\Psi\| \equiv \sum_{X \ni 0} |X|^{-1} \|\psi_x\| < \infty$. Then there exists a function $F(m) = F(m, \Psi)$ with $\lim_{m\to\infty} F(m) = 0$, such that*

$$\limsup_{n\to\infty} \frac{1}{|\Lambda(n)|} \left\| K_{\Lambda(n)} - \sum_{j=1}^{k^d} K_{C_j} \right\| \le F(m). \tag{4.1}$$

We will also use an immediate consequence of Lemma 4.1.

Corollary 4.2. *Let Ψ be an interaction with $\|\Psi\| < \infty$. Then there exists a function $F(m) = F(m, \Psi)$ with $\lim_{m\to\infty} F(m) = 0$, such that*

$$\limsup_{n\to\infty} \left\| \frac{1}{|\Lambda(n)|} K_{\Lambda(n)} - \frac{1}{|\Lambda(km)|} \sum_{j=1}^{k^d} K_{C_j} \right\| \le F(m). \tag{4.2}$$

Proof of Lemma 4.1. To simplify notation we set $l = m + 2g(m)$ in the proof. If $D = \{x \in \mathbb{Z}^d; a_i \le x_i < a_i + l\}$ is a cube of side length l and $r \in \mathbb{N}$ such that $r < l/2$ we denote $D_r = \{x \in \mathbb{Z}^d; a_i + r \le x_i < a_i + l - r\}$ the cube of side length $l - 2r$ centered at the same point as D.

Let us consider two cubes $D \subset D' \subset \mathbb{Z}^d$. We have

$$\begin{aligned}
\|K_{D'} - K_D\| &\le \sum_{\substack{X \subset D' \\ X \not\subset D}} \|\psi_X\| \le \sum_{x \in D'} \sum_{\substack{X \ni x \\ X \not\subset D}} \frac{1}{|X|} \|\psi_X\| \\
&\le \sum_{x \in D' \setminus D_r} \sum_{X \ni x} \frac{1}{|X|} \|\psi_X\| + \sum_{x \in D_r} \sum_{\substack{X \ni x \\ X \not\subset D}} \frac{1}{|X|} \|\psi_X\| \\
&\le |D' \setminus D_r| \|\Psi\| + |D_r| \sum_{\substack{X \ni 0 \\ \mathrm{diam}(X) > r}} \frac{1}{|X|} \|\psi_X\|.
\end{aligned} \tag{4.3}$$

Using (4.3), we have for any r,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|\Lambda(n)|} \|K_{\Lambda(n)} - K_{\Lambda(kl)}\| \\
&\quad \le \lim_{n\to\infty} \left[\frac{|\Lambda(n) \setminus \Lambda_r(kl)|}{|\Lambda(n)|} \|\Psi\| + \frac{|\Lambda_r(kl)|}{|\Lambda(n)|} \sum_{\substack{X \ni 0 \\ \mathrm{diam}(X) > r}} \frac{1}{|X|} \|\psi_X\| \right] \\
&\quad = \sum_{\substack{X \ni 0 \\ \mathrm{diam}(X) > r}} \frac{1}{|X|} \|\psi_X\|.
\end{aligned}$$

Since r is arbitrary, we have

$$\limsup_{n\to\infty} \frac{1}{|\Lambda(n)|}\|K_{\Lambda(n)} - K_{\Lambda(kl)}\| = 0. \tag{4.4}$$

Using (4.3) again, we have

$$\limsup_{n\to\infty} \frac{1}{|\Lambda(n)|}\left\|\sum_{j=1}^{k^d}(K_{\tilde{C}_j} - K_{C_j})\right\|$$
$$\leq \lim_{n\to\infty} \frac{k^d|\Lambda(l)|}{|\Lambda(n)|}\left[\frac{|\Lambda(l)\backslash\Lambda_r(m)|}{|\Lambda(l)|}\|\Psi\| + \frac{|\Lambda_r(m)|}{|\Lambda(l)|}\sum_{\substack{X\ni 0\\ \mathrm{diam}(X)>r}}\frac{1}{|X|}\|\psi_X\|\right].$$

If $r = h(m)$ with $\lim_{m\to\infty} h(m) = \infty$ and $\lim_{m\to\infty} h(m)/m = 0$, we get

$$\limsup_{n\to\infty} \frac{1}{|\Lambda(n)|}\left\|\sum_{j=1}^{k^d}(K_{\tilde{C}_j} - K_{C_j})\right\| = o(m). \tag{4.5}$$

Finally

$$\left\|K_{\Lambda(kl)} - \sum_{j=1}^{k^d} K_{\tilde{C}_j}\right\| \leq \sum_{\substack{X\subset\Lambda(kl)\\ X\not\subset \mathrm{some}\tilde{C}_j}}\|\psi_X\| = \sum_{\substack{X\subset\Lambda(kl)\\ X\not\subset \mathrm{some}\tilde{C}_j}}\sum_{j=1}^{k^d}\frac{|X\cap\tilde{C}_j|}{|X|}\|\psi_X\|$$
$$\leq |\Lambda(kl)|\frac{1}{k^d}\sum_{j=1}^{k^d}\frac{1}{|\tilde{C}_j|}\sum_{X\not\subset\tilde{C}_j}\frac{|X\cap\tilde{C}_j|}{|X|}\|\psi_X\| = |\Lambda(kl)|d(\Psi,l) \tag{4.6}$$

with

$$d(\Psi,l) = \frac{1}{|\Lambda(l)|}\sum_{X\not\subset\Lambda(l)}\frac{|X\cap\Lambda(l)|}{|X|}\|\psi_X\| = \sum_{x\in\Lambda(l)}\sum_{\substack{X\ni x\\ X\not\subset\Lambda(l)}}\frac{1}{|X||\Lambda(l)|}\|\psi_X\|$$
$$\leq \frac{|\Lambda_r(l)|}{|\Lambda(l)|}\sum_{\substack{X\ni 0\\ \mathrm{diam}(X)>r}}\frac{1}{|X|}\|\psi_X\| + \frac{|\Lambda(l)| - |\Lambda_r(l)|}{|\Lambda(l)|}\|\Psi\|. \tag{4.7}$$

Since $l = m + 2g(m)$ if we pick $r = h(m)$ as above we get

$$\limsup_{n\to\infty} \frac{1}{|\Lambda(n)|}\left\|K_{\Lambda(kl)} - \sum_{j=1}^{k^d} K_{\tilde{C}_j}\right\| = o(m). \tag{4.8}$$

Combining the bounds (4.4), (4.5) and (4.8) concludes the proof of Lemma 4.1. □

Proof of Corollary 4.2. An easy estimate shows that the difference between $\||\Lambda(n)|^{-1}K_{\Lambda(n)} - |\Lambda(km)|^{-1}\sum_{j=1}^{k^d} K_{C_j}\|$ and $|\Lambda(n)|^{-1}\|K_{\Lambda(n)} - \sum_{j=1}^{k^d} K_{C_j}\|$ is $O(g(m)/m)\|\Psi\|$. □

The second fact is a general remark on the strategy to prove the existence of a concave RL function [23].

Remark 4.3. Let x, x_1, x_2 such that $\frac{1}{2}(x_1 + x_2) = x$ and let $0 < \varepsilon' < \varepsilon$. To prove the existence of a concave RL-function it is enough to prove that

$$\underline{m}(B_\varepsilon(x)) \geq \frac{\overline{m}(B_{\varepsilon'}(x_1)) + \underline{m}(B_{\varepsilon'}(x_2))}{2}. \tag{4.9}$$

Indeed if we set $x_1 = x_2 = x$ in (4.9), then we obtain

$$\underline{s}(x) \geq \overline{s}(x),$$

and therefore the Ruelle–Lanford function $s(x)$ exists. Using then (4.9) again, we obtain that

$$s(x) \geq \frac{s(x_1) + s(x_2)}{2}.$$

Since $s(x)$ is upper-semicontinuous, this implies that $s(x)$ is concave.

4.2. *Tracial state and conserved quantities*

In this section, we prove a quantum large deviation theorem in the simplest possible case. We bypass a number of issue associated to taking thermodynamic limits for the states by considering first the *finite volume Gibbs states*

$$\omega_{\Lambda(n)}(A) = \frac{\mathrm{tr}(Ae^{-H_{\Lambda(n)}})}{\mathrm{tr}(e^{-H_{\Lambda(n)}})}.$$

In addition, we assume that the Hamiltonian and that the macroscopic observables K_Λ is a *conserved quantity*, i.e. the commutators $[K_\Lambda, H_\Lambda]$ vanish for all Λ. Note that, although very restrictive, this condition is, in general, satisfied for thermodynamic quantities such as magnetization, density, energy, etc. The following theorem provides a (weak) justification that macroscopic conserved quantities are exponentially concentrated in equilibrium.

An important special case is the case where $H_\Lambda = 0$, that is one consider the tracial state tr. In this case any observable K_Λ can be chosen arbitrarily and the rate function $s(x)$ is the microcanonical entropy whose existence is of course well known. The large deviation statement for the tracial state can be found, e.g., in [36]; the only novelty here, maybe, is a very simple proof.

Theorem 4.4. *Let Φ and Ψ be interaction with $\|\Phi\| < \infty$ and $\|\Psi\| < \infty$. Suppose that the commutators $[K_{\Lambda(n)}, H_{\Lambda(n)}]$ commute for all n. Then the probability measures*

$$\mu_n(A) = \frac{\mathrm{tr}(\mathbf{I}_A(|\Lambda(n)|^{-1}K_{\Lambda(n)})e^{-H_\Lambda(n)})}{\mathrm{tr}(e^{-H_\Lambda(n)})},$$

satisfies a large deviation principle on the scale $|\Lambda(n)|$ with a concave rate function $s(x)$. We have

$$\sup_x(\alpha x + s(x)) = P(\alpha), \quad s(x) = \inf_\alpha (P(\alpha) - \alpha x),$$

where $P(\alpha) = \lim_{n\to\infty} |\Lambda(n)|^{-1} \log \mathrm{tr}(e^{-H_{\Lambda(n)}+\alpha K_{\Lambda(n)}})$ is the translated free energy.

Proof. Let us choose x, x_1, x_2 and ε, ε' as in Remark 4.3. Given $n > m$ let k be the even integer such that $n = km + r$ with $0 \leq r < 2m - 1$ (having k even is useful later). Divide the cube $\Lambda(km)$ into k^d disjoint contiguous cube $C_j, j = 1, \ldots, k^d$ each of which is a translate of the cube $\Lambda(m)$.

Let us denote by $\lambda_j^{(n)}$ the eigenvalues of $H_{\Lambda(n)}$ and by $\mu_j^{(n)}$ the eigenvalues of $K_{\Lambda(n)}$. Since $H_{\Lambda(n)}$ and $K_{\Lambda(n)}$ commute we have

$$\mu_n(B_\varepsilon(x)) = \frac{\displaystyle\sum_{j;\frac{\mu_j^{(n)}}{|\Lambda(n)|}\in B_\varepsilon(x)} e^{-\lambda_j^{(n)}}}{\displaystyle\sum_j e^{-\lambda_j^{(n)}}}. \tag{4.10}$$

By Corollary 4.2, we can choose M and $N = N_m$ so that for $m > M$ and $n > N$ we have

$$\left\| |\Lambda(n)|^{-1} K_{\Lambda(n)} - |\Lambda(km)|^{-1} \sum_{j=1}^{k^d} K_{C_j} \right\| \leq (\varepsilon - \varepsilon').$$

Let $\mu^{(m)}$ be an eigenvalue of $K_{\Lambda(m)}$ with $\mu^{(m)}/|\Lambda(m)| \in B_{\varepsilon'}(x_1)$ and let $\hat\mu^{(m)}$ be an eigenvalue of $K_{\Lambda(m)}$ with $\hat\mu^{(m)}/|\Lambda(m)| \in B_{\varepsilon'}(x_2)$. Let us assign $\mu^{(m)}$ to each cube C_j with $j = 1, \ldots, \frac{k^d}{2}$ and $\hat\mu^{(m)}$ to the each cube C_j with $j = \frac{k^d}{2} + 1, \ldots, k^d$. Then $\tilde\mu^{(km)} \equiv \frac{k^d}{2}(\mu^{(m)} + \hat\mu^{(m)})$ is an eigenvalue of $\sum_j K_{C_j}$ such that $\tilde\mu^{(km)}/|\Lambda(km)| \in B_{\varepsilon'}(x)$. For $m > M$ and $n \geq N = N_m$, by Weyl's perturbation theorem, for any choice of $\mu^{(m)}$ and $\hat\mu^{(m)}$ there exists an eigenvalue $\mu^{(n)}$ of $K_\Lambda(n)$ such that $\mu^{(n)}/|\Lambda(n)| \in B_\varepsilon(x)$.

Assume that the eigenvalues $\lambda_i^{(n)}$ of $H_{\Lambda(n)}$ are listed in increasing order, counting multiplicity. Let $\tilde\lambda_i^{(n)}$ be the eigenvalues of $\sum_j H_{C_j} \otimes 1_{\Lambda(n)\setminus\Lambda(km)}$ also listed in increasing order. By Weyl's perturbation theorem, and Lemma 4.1, there exists M' such that for $m > M'$ there exists $N' = N'_m$ such that $n \geq N'$ we have

$$\tilde\lambda_i^{(n)} - |\Lambda(n)|F(m) \leq \lambda_i^{(n)} \leq \tilde\lambda_i^{(n)} + |\Lambda(n)|F(m).$$

Using the formula (4.10), we obtain that

$$\mu_n(B_\varepsilon(x)) \geq \mu_m(B_{\varepsilon'}(x_1))^{\frac{k^d}{2}} \mu_m(B_{\varepsilon'}(x_2))^{\frac{k^d}{2}} e^{-2|\Lambda(n)|F(m)}$$

and thus

$$\frac{\log \mu_n(B_\varepsilon(x))}{|\Lambda(n)|} \geq \left(\frac{\log \mu_m(B_{\varepsilon'}(x_1))}{2|\Lambda(m)|} + \frac{\log \mu_m(B_{\varepsilon'}(x_2))}{2|\Lambda(m)|}\right) \frac{k^d|\Lambda(m)|}{|\Lambda(n)|} - 2F(m).$$

To conclude we take first a lim inf over n keeping m fixed and then choose a subsequence m_l such that $\lim_{l\to\infty} |\Lambda(m_l)|^{-1} \log \mu_{m_l}(B_{\varepsilon'}(x_1)) = \overline{m}(B'_\varepsilon(x_1))$. Together with Remark 4.3 this concludes the proof of Theorem 4.4. □

Theorem 4.5. *Let Φ and Ψ be interactions with $\|\Phi\| < \infty$ and $\|\Psi\| < \infty$. Suppose that the commutators $[K_{\Lambda(n)}, H_{\Lambda(n)}]$ vanish for all n. Suppose $\omega^{(\Phi)}$ satisfies the condition (3.1). Then the probability measure*

$$\mu_n(A) = \omega^{(\Phi)}(\mathbf{I}_A(|\Lambda(n)|^{-1} K_{\Lambda(n)}))$$

satisfies a large deviation principle on the scale $|\Lambda(n)|$ with a concave rate function $s(x)$. We have

$$\sup_x(\alpha x + s(x)) = P(\alpha), \quad s(x) = \inf_\alpha (P(\alpha) - \alpha x),$$

where $P(\alpha) = \lim_{n\to\infty} |\Lambda(n)|^{-1} \log \operatorname{tr}(e^{-H_{\Lambda(n)} + \alpha K_{\Lambda(n)}})$ is the translated free energy.

Proof. Since

$$\omega^{(\Phi)}(\mathbf{I}_A(|\Lambda(n)|^{-1} K_{\Lambda(n)})) \geq e^{-c(n)} \frac{\operatorname{tr}(\mathbf{I}_A(|\Lambda(n)|^{-1} K_{\Lambda(n)}) e^{-H_{\Lambda}(n)})}{\operatorname{tr}(e^{-H_{\Lambda}(n)})},$$

the theorem follows immediately from Theorem 4.4. □

Remark 4.6 (Equivalence of Ensembles). For the tracial case it is not difficult [36] to show the variational formula $s(x) = \sup\{s(\omega); \omega(A_\Psi) = x\}$ where $s(\omega)$ is the specific entropy of the state ω and that the supremum is attained exactly if $\omega = \omega^{\beta\Phi}$ is a Gibbs-KMS state at temperature $\beta = \beta(x)$ with β chosen in such a way that $\omega^{\beta\Psi}(A_\Psi) = x$. This is the equivalence of ensemble: the thermodynamic function entropy can be computed via microcanonical or canonical prescriptions. Furthermore, the LDP can be used to prove that suitable microcanonical states are equivalent to canonical states, see [36] for the classical case and [24, 25] for the quantum case. Non-commutative versions of equivalence of ensembles are considered in [7].

4.3. *Classical subalgebras*

In this section, we assume that ω is an asymptotically decoupled state and that $\Psi \in \mathcal{B}$ is a classical interaction, i.e. there exists a classical subalgebra $\mathcal{O}^{(cl)} \subset \mathcal{O}$ such that, for all X, $\psi_X \in \mathcal{O}^{(cl)}$. For example if $\Psi = \{\psi_x\}_{x\in\mathbb{Z}^d}$ consists of only of "one-site" interactions then Ψ is classical. More generally any classical spin system is described by a classical interaction. Note that we do not assume any relation between the interaction Ψ and the state ω; if $\omega = \omega^\Phi$ is a Gibbs state for the interaction Φ then Φ and Ψ need not commute.

As noted in Sec. 3.1 the restriction of ω on $\mathcal{O}^{(cl)}$ can be identified with a probability measure $d\omega^{(cl)}$ on the configuration space $\mathcal{L}$. Furthermore, it is easy to see that the state $\omega^{(cl)}$ on the C^*-algebra $\mathcal{O}^{(cl)} \simeq C(\mathcal{L})$ is asymptotically decoupled whenever the state ω on $\mathcal{O}$ is asymptoticallly decoupled.

We have

Theorem 4.7. *Let Ψ be a classical interaction with $\|\Psi\| < \infty$ and let ω be an asymptotically decoupled state. Then the sequence of probability measures*

$$\mu_n(A) = \omega(\mathbf{I}_A(|\Lambda(n)|^{-1}K_{\Lambda(n)})),$$

satisfies a large deviation principle on the scale $|\Lambda(n)|$ with a concave rate function $s(x)$. Moreover

$$s(x) = \inf_{\alpha}(f(\alpha) - \alpha x),$$

where

$$f(\alpha) = \lim_{n\to\infty} \frac{1}{|\Lambda(n)|} \log \omega(\exp(\alpha K_{\Lambda(n)})).$$

Proof. The proof reduces to the classical case (see [23]) since the measures μ_n can be written as

$$\mu_n(A) = \omega^{(cl)}(\mathbf{I}_A(|\Lambda(n)|^{-1}K_{\Lambda(n)})) = \int \mathbf{I}_{\{|\Lambda(n)|^{-1}K_{\Lambda(n)}\in A\}}(l)d\omega^{(cl)}(l)$$

and the restriction of $\omega^{(cl)}$ on $\mathcal{O}^{(cl)}$ is asymptotically decoupled. Following Remark 4.3 we choose arbitrary x, x_1, x_2 such that $\frac{x_1}{2} + \frac{x_2}{2} = x$ and $0 < \varepsilon' < \varepsilon$. We divide the cube $\Lambda(n)$ as explained before Lemma 4.1. We choose M and $N = N_m$ such that for $m > M$ and $n > N$

$$\left\| \frac{1}{|\Lambda(n)|}K_{\Lambda(n)} - \frac{1}{|\Lambda(km)|}\sum_{j=1}^{k^d} K_{C_j} \right\| \le \varepsilon - \varepsilon'. \tag{4.11}$$

Let l_{C_j} be configurations such that $K_{C_j}(l_{C_j})/|C_j| \in B_{\varepsilon'}(x_1)$ for $1 \le j \le \frac{k^d}{2}$ and $K_{C_j}(l_{C_j})/|C_j| \in B_{\varepsilon'}(x_2)$ for $\frac{k^d}{2} + 1 \le j \le k^d$. By (4.11) any configuration $l_{\Lambda(n)}$ which coincides with l_{C_j} on all C_j satisfies $K_{\Lambda(n)}(l_{\Lambda(n)})/|\Lambda(n)| \in B_\varepsilon(x)$.

Therefore using the fact that $\omega^{(cl)}$ is asymptotically decoupled we have the bound

$$\begin{aligned}
&\omega\left(\mathbf{I}_{B_\varepsilon(x)}\left(\frac{K_{\Lambda(n)}}{|\Lambda(n)|}\right)\right)\\
&\quad = \int \mathbf{I}_{\left\{\frac{K_{\Lambda(n)}}{|\Lambda(n)|}\in B_\varepsilon(x)\right\}} d\omega^{(cl)}\\
&\quad \ge \int \prod_{j=1}^{\frac{k^d}{2}} \mathbf{I}_{\left\{\frac{K_{C_j}}{|C_j|}\in B_{\varepsilon'}(x_1)\right\}} \prod_{\frac{k^d}{2}+1}^{k^d} \mathbf{I}_{\left\{\frac{K_{C_j}}{|C_j|}\in B_{\varepsilon'}(x_2)\right\}} d\omega^{(cl)}\\
&\quad \ge \left(\int \mathbf{I}_{\left\{\frac{K_{\Lambda(m)}}{|\Lambda(m)|}\in B_{\varepsilon'}(x_1)\right\}} d\omega^{(cl)}\right)^{\frac{k^d}{2}} \left(\int \mathbf{I}_{\left\{\frac{K_{\Lambda(m)}}{|\Lambda(m)|}\in B_{\varepsilon'}(x_2)\right\}} d\omega^{(cl)}\right)^{\frac{k^d}{2}} e^{-c(m)k^d}.
\end{aligned}$$

Thus we obtain

$$\frac{\log \mu_n(B_\varepsilon(x))}{|\Lambda(n)|} \geq \left(\frac{\log \mu_m(B_{\varepsilon'}(x_1))}{2|\Lambda(m)|} + \frac{\log \mu_m(B_{\varepsilon'}(x_2))}{2|\Lambda(m)|}\right)\frac{k^d|\Lambda(m)|}{|\Lambda(n)|} - \frac{1}{|\Lambda(n)|}c(m)k^d.$$

We conclude by taking the $\liminf$ over n and then choosing a subsequence m_l such that $\lim_{l\to\infty}(|\Lambda(m_l)|)^{-1}\log \mu_{m_l}(B_{\varepsilon'}(x_1)) = \overline{m}(B'_\varepsilon(x_1))$. The identification of the rate function follows from Varadhan's lemma. □

Remark 4.8. One can show (see [32, 29] for more details) that the rate function satisfies the following variational characterization:

$$s(x) = \sup\{-h_{cl}(\nu, \omega^{(cl)}); \nu(A_\Psi) = x\},$$

where h_{cl} is the classical relative entropy per unit volume, and the supremum is taken over all classical translation invariant states.

4.4. *Dimension* 1

Throughout this section we assume that $d = 1$ (so we write $|\Lambda(n)| = n$) and that ω is an asymptotically decoupled state, for example we may assume that ω a KMS-Gibbs state for a finite range interaction. We also assume that Ψ is a *finite range* interaction.

The crucial estimate needed to control the effect of non-commutativity is an estimate on the difference between the spectral projections associated to $K_{\Lambda(n)}$ and $\sum_{j=1}^k K_{C_j}$ (see Sec. 4.1). To prove this we relies on a "cocycle estimate" proved in [1], which follows from the fact that the time-evolution $\tau_t(A)$ of any local observable A for a finite-range quantum spin system can be extended to a entire analytic function of t. This allows to prove the following "exponential version" of Lemma 4.1.

Proposition 4.9. *Let Ψ be a finite range interaction of range R and let $\beta \in \mathbb{R}$. Then there exists a function $F_\beta(m) = F_\beta(m, R, \Psi)$ with*

$$\lim_{m\to\infty} F_\beta(m) = 0. \tag{4.12}$$

such that

$$\limsup_{n\to\infty} \frac{1}{n}\log\left\|e^{\beta K_{\Lambda(n)}}e^{-\beta\sum_{j=1}^k K_{C_j}}\right\| \leq |\beta|F_\beta(m). \tag{4.13}$$

Proof. The proof is an application of the results in [1], see in particular Secs. 4 and 5. The basic bound in [1, Sec. 5], is that if $A_X \in \mathcal{O}_X$ with $\mathrm{diam}(X) \leq R$ then there exists a constant $D(\beta, R, \Psi)$ such that

$$\|e^{\beta K_\Lambda(n)}e^{-\beta(K_\Lambda(n) - A_X)}\| \leq e^{|\beta|D(\beta,R,\Psi)\|A_X\|}. \tag{4.14}$$

The bound (4.14) follows from Dyson formula and estimates (uniform in n) on the dynamics in imaginary time generated by the Hamiltonian $K_{\Lambda(n)}$. To apply these results here we write

$$K_{\Lambda(n)} = \sum_{j=1}^{k} K_{C_j} + \sum_{\substack{X\subset\Lambda(n)\\ X\not\subset \mathrm{some} C_j}} \psi_X.$$

Let $t_X \in \{0,1\}$ and let us define the family of interpolating Hamiltonians

$$K_{\Lambda(n)}(\{t_X\}) = \sum_{j=1}^{k} K_{C_j} + \sum_{\substack{X\subset\Lambda(n)\\ X\not\subset \mathrm{some} C_j}} t_X\psi_X.$$

The estimates on the dynamics in [1] are easily seen to be uniform in $\{t_X\}$ and so we can apply the bound (4.14) iteratively, changing at each step one t_X from 1 to 0. Using that Ψ has a finite range R we obtain the bound

$$\|e^{\beta K_{\Lambda(n)}} e^{-\beta\sum_{j=1}^{k} K_{C_j}}\| \le e^{|\beta| D(\beta,R,\Psi)\sum_{\substack{X\subset\Lambda(n)\\ X\not\subset \mathrm{some} C_j}} \|\psi_X\|}.$$

But the sum over X is now treated exactly as Lemma 4.1 and we find $F_\beta(m) = F(m)D(\beta,R,\Psi)$. □

We use this bound to prove an exponential estimates which control how the spectral projections change when we replace $K_{\Lambda(n)}$ by $\sum_{j=1}^{k} K_{C_j}$.

Proposition 4.10. *Let $\varepsilon > \varepsilon' > 0$. Then for any $\alpha > 0$ there exists a function $\tilde{F}_\alpha(m)$ with $\lim_{m\to\infty}\tilde{F}_\alpha(m) = 0$ such that*

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log&\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k} K_{C_j}\right)\mathbf{I}_{B_\varepsilon(x)^C}(n^{-1}K_{\Lambda(n)})\right\|\\ &\le -\alpha(\varepsilon-\varepsilon'-\tilde{F}_\alpha(m)).\end{aligned} \tag{4.15}$$

Proof. Let us write

$$K_{\Lambda(n)} = \sum_i \mu_i P_i, \quad \sum_{j=1}^{k} K_{C_j} = \sum_l \lambda_l Q_l, \tag{4.16}$$

where P_i and Q_l are rank-one projections and μ_i and λ_l are the eigenvalues of $K_{\Lambda(n)}$ and $\sum_j K_{C_j}$. For any $\beta\in\mathbb{R}$

$$\begin{aligned}\mathbf{I}_{B_\delta(y)}(n^{-1}K_{\Lambda(n)}) &= \sum_{i;\frac{\mu_i}{n}\in B_\delta(y)} P_i\\ &= e^{\beta(K_{\Lambda(n)}-ny)}\sum_{i;\frac{\mu_i}{n}\in B_\delta(y)} e^{-\beta(\mu_i-ny)}P_i\\ &\equiv e^{\beta(K_{\Lambda(n)}-ny)}V_{\beta,y,\delta}\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k}K_{C_j}\right) &= \sum_{l;\frac{\lambda_l}{mk}\in B_{\varepsilon'}(x)} Q_l \\ &= \sum_{l;\frac{\lambda_l}{mk}\in B_{\varepsilon'}(x)} e^{\beta(\lambda_l - xn)}Q_l e^{-\beta(\sum_j K_{C_j}-xn)} \\ &\equiv W_{\beta,x,\varepsilon'}e^{-\beta(\sum_j K_{C_j}-xn)}, \end{aligned}\tag{4.18}$$

with the bounds

$$\|V_{\beta,y,\delta}\| \le e^{|\beta|n\delta}, \quad \|W_{\beta,x,\varepsilon'}\| \le e^{|\beta|mk(\varepsilon'+(\frac{n}{mk}-1)|x|)}.\tag{4.19}$$

If $y > x$ we choose $\beta = \alpha > 0$ and using Eqs. (4.17) and (4.18) as well as the bounds (4.13) and (4.19), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k}K_{C_j}\right)\mathbf{I}_{B_\delta(y)}(n^{-1}K_{\Lambda(n)})\right\| \\ &\quad= \limsup_{n\to\infty}\frac{1}{n}\log\|W_{\alpha,x,\varepsilon'}e^{-\alpha(\sum_j K_{C_j}-nx)}e^{\alpha(K_{\Lambda(n)}-ny)}V_{\alpha,y,\delta}\| \\ &\quad\le \limsup_{n\to\infty}\Bigg[-\alpha(y-x)+\frac{1}{n}\log\|e^{-\alpha\sum_j K_{C_j}}e^{\alpha K_{\Lambda(n)}}\| \\ &\qquad + \frac{\alpha}{n}(n\delta+mk\varepsilon'+(n-mk)|x|)\Bigg] \\ &\quad\le -\alpha(y-x)+\alpha F_\alpha(m)+\alpha(\delta+\varepsilon')+\alpha\frac{g(m)}{m}|x|.\end{aligned}$$

Similarly, for $y < x$, we choose $\beta = -\alpha$ and obtain a similar bound and finally

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k}K_{C_j}\right)\mathbf{I}_{B_\delta(y)}(n^{-1}K_{\Lambda(n)})\right\| \\ &\qquad\le -\alpha|y-x|+\alpha F_\alpha(m)+\alpha(\delta+\varepsilon')+\alpha\frac{g(m)}{m}|x|.\end{aligned}\tag{4.20}$$

Next we choose δ be such that $\varepsilon > 2\delta + \varepsilon'$ and choose finitely many intervals T_l and $x_l \in T_l$, $l = 1,\dots,L$ such that

$$B_\varepsilon(x)^C \cap [-\|\Psi\|,\|\Psi\|] = \bigcup_l T_l, \quad T_l \subset B_\delta(x_l).$$

By the principle of the largest term, and using the bound (4.20), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k}K_{C_j}\right)\mathbf{I}_{B_\varepsilon(x)^C}(n^{-1}K_{\Lambda(n)})\right\| \\
&\quad\le \limsup_{n\to\infty}\frac{1}{n}\log\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k}K_{C_j}\right)\sum_{l=1}^{L}\mathbf{I}_{T_l}(n^{-1}K_{\Lambda(n)})\right\| \\
&\quad\le \max_l\limsup_{n\to\infty}\frac{1}{n}\log\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left((mk)^{-1}\sum_{j=1}^{k}K_{C_j}\right)\mathbf{I}_{B_\delta(x_l)}(n^{-1}K_{\Lambda(n)})\right\| \\
&\quad\le -\alpha(\varepsilon-\varepsilon'-\delta)+\alpha\left(F_\alpha(m)+\frac{g(m)}{m}|x|\right). \qquad (4.21)
\end{aligned}$$

Since δ is arbitrary this concludes the proof with $\tilde{F}_\alpha(m) = F_\alpha(m) + \frac{g(m)}{m}|x|$. □

With this estimate we can now prove

Theorem 4.11. *Let $d = 1$, let ω be an asymptotically decoupled translation invariant state, and let Ψ be a finite range interaction. Then the sequence of probability measures*

$$\mu_n(A) = \omega(\mathbf{I}_A(n^{-1}K_{\Lambda(n)})),$$

satisfies a large deviation principle with a concave rate function $s(x)$. Moreover

$$s(x) = \inf_\alpha(f(\alpha) - \alpha x),$$

where

$$f(\alpha) = \lim_{n\to\infty} n^{-1}\log\omega(\exp(\alpha K_{\Lambda(n)})).$$

Proof. Let ω be an asymptotically decoupled state with parameters g and c. Let x, x_1, x_2 be such that $\frac{x_1}{2} + \frac{x_2}{2} = x$ and $0 < \varepsilon' < \varepsilon$. For any $n > m$ we decompose $\Lambda(n)$ as in Sec. 4.1. Note that

$$\bigotimes_{j=1}^{k/2}\mathbf{I}_{B_{\varepsilon'}(x_1)}\left(\frac{K_{C_j}}{m}\right)\bigotimes_{j=k/2+1}^{k}\mathbf{I}_{B_{\varepsilon'}(x_2)}\left(\frac{K_{C_j}}{m}\right) \le \mathbf{I}_{B_{\varepsilon'}(x)}\left(\frac{\sum_j K_{C_j}}{mk}\right), \qquad (4.22)$$

and that for any projections P and Q and a state ω we have

$$\begin{aligned}
\omega(P) &= \omega(QPQ) + \omega((1-Q)PQ + QP(1-Q)) + \omega((1-Q)P(1-Q)) \\
&\le \omega(Q) + 2\|(1-Q)PQ\| + \|(1-Q)P(1-Q)\| \\
&\le \omega(Q) + 3\|(1-Q)P\|. \qquad (4.23)
\end{aligned}$$

Using that ω is asymptotically decoupled, and estimates (4.22) and (4.23), we obtain

$$
\begin{aligned}
&\frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_1)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)+\frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_2)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)\\
&\quad\le \frac{1}{mk}\log\omega\left(\bigotimes_{j=1}^{k/2}\mathbf{I}_{B_{\varepsilon'}(x_1)}\left(\frac{K_{C_j}}{m}\right)\bigotimes_{j=k/2+1}^{k}\mathbf{I}_{B_{\varepsilon'}(x_2)}\left(\frac{K_{C_j}}{m}\right)\right)+\frac{c(m)k}{mk}\\
&\quad\le \frac{1}{mk}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x)}\left(\frac{\sum_j K_{C_j}}{mk}\right)\right)+\frac{c(m)}{m}\\
&\quad\le \frac{1}{mk}\log\left[\omega\left(\mathbf{I}_{B_{\varepsilon}(x)}\left(\frac{K_{\Lambda(n)}}{n}\right)\right)\right.\\
&\qquad\left.+3\left\|\mathbf{I}_{B_{\varepsilon'}(x)}\left(\frac{\sum_{j=1}^{k}K_{C_j}}{mk}\right)\mathbf{I}_{B_{\varepsilon}(x)^C}\left(\frac{K_{\Lambda(n)}}{n}\right)\right\|\right]+\frac{c(m)}{m}.
\end{aligned}
$$

Keeping m fixed we take a $\liminf$ over n and using Proposition 4.10 we obtain

$$
\begin{aligned}
&\frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_1)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)+\frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_2)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)\\
&\quad\le\left(1+\frac{g(m)}{m}\right)\max\{\underline{m}(B_\varepsilon(x)),-\alpha(\varepsilon-\varepsilon'-\tilde{F}_\alpha(m))\}+\frac{c(m)}{m}.
\end{aligned}
\tag{4.24}
$$

To conclude we will use the bound (4.24) repeteadly.

(a) Assume first $x=x_1=x_2$ and assume that $\underline{s}(x)>-\infty$. Choose first α so large that

$$-\frac{1}{2}\alpha(\varepsilon-\varepsilon')<\underline{m}(B_\varepsilon(x)),$$

and then $M=M(\alpha)$ so that $\tilde{F}_\alpha(m)\le\frac{1}{2}(\varepsilon-\varepsilon')$ for $m>M$. By (4.24) we have then

$$\frac{1}{m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)\le\left(1+\frac{g(m)}{m}\right)\underline{m}(B_\varepsilon(x))+\frac{c(m)}{m},$$

and thus $\overline{m}(B_{\varepsilon'}(x))\le\underline{m}(B_\varepsilon(x))$. This implies that the Ruelle function $s(x)$ exists and is finite.

(b) Assume that $\underline{s}(x) > -\infty$ and $x = \frac{1}{2}(x_1 + x_2)$. Repeating the same argument as in (a) one obtains, for m large enough,

$$\frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_1)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right) + \frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_2)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)$$
$$\leq \left(1 + \frac{g(m)}{m}\right)\underline{m}(B_\varepsilon(x)) + \frac{c(m)}{m},$$

and this implies that $\frac{1}{2}\overline{m}(B_{\varepsilon'}(x_1)) + \frac{1}{2}\underline{m}(B_{\varepsilon'}(x_2)) \leq \underline{m}(B_\varepsilon(x))$. Thus the rate function $s(x)$ is concave wherever it is finite.

(c) Let us assume that $\underline{s}(x) = -\infty$. Then for any $t > 0$ we can find ε_t such that for $\varepsilon < \varepsilon_t$ we have $\underline{m}(B_\varepsilon(x)) \leq -t$. By (4.24) we have

$$\frac{1}{m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)$$
$$\leq \left(1 + \frac{g(m)}{m}\right)\max\{-t, -\alpha(\varepsilon - \varepsilon' - \tilde{F}_\alpha(m))\} + \frac{c(m)}{m},$$

and thus taking $m \to \infty$ we obtain

$$\overline{m}(B_{\varepsilon'}(x)) \leq \max\{-t, -\alpha(\varepsilon - \varepsilon')\},$$

and so

$$\overline{s}(x) \leq \max\{-t, -\alpha\varepsilon\}.$$

Since α and t are arbitrary we have $\overline{s}(x) = -\infty$.

(d) Assume that $\underline{s}(x) = -\infty$ and $x = \frac{1}{2}(x_1 + x_2)$. Repeating the same argument as in (c) for any $t > 0$ there exists $\varepsilon_t > 0$ such that for all $\alpha > 0$,

$$\frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_1)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right) + \frac{1}{2m}\log\omega\left(\mathbf{I}_{B_{\varepsilon'}(x_2)}\left(\frac{K_{\Lambda(m)}}{m}\right)\right)$$
$$\leq \left(1 + \frac{g(m)}{m}\right)\max\{-t, -\alpha(\varepsilon_t - \varepsilon' - \tilde{F}_\alpha(m))\} + \frac{c(m)}{m}$$

and this implies that $\frac{1}{2}\overline{m}(B_{\varepsilon'}(x_1)) + \frac{1}{2}\underline{m}(B_{\varepsilon'}(x_2)) \leq \max\{-t, -\alpha(\varepsilon_t - \varepsilon')\}$. Hence we obtain

$$\frac{1}{2}s(x_1) + \frac{1}{2}s(x_2) = \frac{1}{2}\overline{s}(x_1) + \frac{1}{2}\underline{s}(x_2) = -\infty \leq s(x).$$

Combining (a)–(d) shows the existence of a concave RL-function and this concludes the proof of Theorem 4.11. ∎

Remark 4.12. A characterization of the rate function using classical relative entropies is proved in [29].

Acknowledgments

The first author was supported by JSPS Grant-in-Aid for Young Scientists (B), Hayashi Memorial Foundation for Female Natural Scientists, Sumitomo Foundation, and Inoue Foundation.

The second author was supported by NSF, Grant DMS-0605058.

References

[1] H. Araki, Gibbs states of a one dimensional quantum lattice, *Comm. Math. Phys.* **14** (1969) 120–157.

[2] H. Araki, On the equivalence of the KMS condition and the variational principle for quantum lattice systems, *Comm. Math. Phys.* **38** (1974) 1–10.

[3] H. Araki and H. Moriya, Equilibrium statistical mechanics of fermion lattice systems, *Rev. Math. Phys.* **15** (2003) 93–198.

[4] I. Bjelakovic, J.-D. Deuschel, T. Kröger, R. Siegmund-Schultze, A. Szkola and R. Seiler, Typical support and Sanov large deviations of correlated states, *Comm. Math. Phys.* **279** (2008) 559–584.

[5] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vols. 1 and 2, Texts and Monographs in Physics (Springer, 1981).

[6] F. Comets, Grandes déviations pour des champs de Gibbs sur $\mathbb{Z}^d$, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986) 511–513.

[7] W. De Roeck, C. Maes and K. Netočny, Quantum macrostates, equivalence of ensembles and an H-theorem, *J. Math. Phys.* **47** (2006) 073303, 12 pp.

[8] W. De Roeck, C. Maes, K. Netočny and L. Rey-Bellet, A note on the noncommutative Laplace–Varadhan integral lemma, *Rev. Math. Phys.* **22**(7) (2010) 839–858.

[9] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Applications of Mathematics, Vol. 38, 2nd edn. (Springer, 1998).

[10] J.-D. Deuschel, D. W. Stroock and H. Zessin, Microcanonical distributions for lattice gases, *Comm. Math. Phys.* **139** (1991) 83–101.

[11] M. Fannes, B. Nachtergaele and R. F. Werner, Finitely correlated states on quantum spin chains, *Comm. Math. Phys.* **144** (1992) 443–490.

[12] H. Föllmer and S. Orey, Large deviations for the empirical field of a Gibbs measure, *Ann. Probab.* **16** (1988) 961–977.

[13] G. Gallavotti, J. L. Lebowitz and V. Mastropietro, Large deviations in rarefied quantum gases, *J. Stat. Phys.* **108** (2002) 831–861.

[14] H.-O. Georgii, Large deviations and maximum entropy principle for interacting random fields on $\mathbb{Z}^d$, *Ann. Probab.* **21** (1993) 1845–1875.

[15] H.-O. Georgii, Large deviations and the equivalence of ensembles for Gibbsian particle systems with superstable interaction, *Probab. Theory Related Fields* **99** (1994) 171–195.

[16] F. Hiai, M. Mosonyi and T. Ogawa, Large deviations and Chernoff bound for certain correlated states on a spin chain, *J. Math. Phys.* **48** (2007) 123301, 19 pp.

[17] F. Hiai, M. Mosonyi, H. Ohno and D. Petz, Free energy density for mean field perturbation of states of a one-dimensional spin chain, *Rev. Math. Phys.* **20** (2008) 335–365.

[18] F. Hiai and D. Petz, Entropy densities for algebraic states, *J. Funct. Anal.* **125** (1994) 287–308.

[19] R. B. Israel, *Convexity in the Theory of Lattice Gases*, Princeton Series in Physics (Princeton University Press, 1979).
[20] O. E. Lanford III, Entropy and equilibrium states in classical statistical mechanics, in *Statistical Mechanics and Mathematical Problems*, Lecture Notes in Physics, Vol. 20 (Springer, 1973), pp. 1–113.
[21] J. L. Lebowitz, M. Lenci and H. Spohn, Large deviations for ideal quantum systems, *J. Math. Phys.* **41** (2000) 1224–1243.
[22] M. Lenci and L. Rey-Bellet, Large deviations in quantum lattice systems: One phase region, *J. Stat. Phys.* **119** (2005) 715–746.
[23] J. T. Lewis, C.-E. Pfister and W. G. Sullivan, Entropy, concentration of probability and conditional limit theorems, *Markov Process. Related Fields* **1** (1995) 319–386.
[24] R. Lima, Equivalence of ensembles in quantum lattice systems, *Ann. Inst. H. Poincaré Sect. A* **15** (1971) 61–68.
[25] R. Lima, Equivalence of ensembles in quantum lattice systems: States, *Comm. Math. Phys.* **24** (1972) 180–192.
[26] T. Matsui, On non-commutative Ruelle transfer operator, *Rev. Math. Phys.* **13** (2001) 1183–1201.
[27] K. Netočny and F. Redig, Large deviations for quantum spin systems, *J. Stat. Phys.* **117** (2004) 521–547.
[28] Y. Ogata, Large deviations in quantum spin chains, *Comm. Math. Phys.* **296** (2010) 35–68.
[29] Y. Ogata and L. Rey-Bellet, The rate function for quantum large deviations, in preparation.
[30] S. Olla, Large deviations for Gibbs random fields, *Probab. Theory Related Fields* **77** (1988) 343–357.
[31] D. Petz, G. A. Raggio and A. Verbeure, Asymptotics of Varadhan-type and the Gibbs variational principle, *Comm. Math. Phys.* **121** (1989) 271–282.
[32] C.-E. Pfister, Thermodynamical aspects of classical lattice systems, in *In and Out of Equilibrium* (*Mambucaba, 2000*), Progr. Probab., Vol. 51 (Birkhäuser, 2002), pp. 393–472.
[33] S. Roelly and H. Zessin, The equivalence of equilibrium principles in statistical mechanics and some applications to large particle systems, *Expo. Math.* **11** (1993) 385–405.
[34] D. Ruelle, Correlation functionals, *J. Math. Phys.* **6** (1965) 201–220.
[35] D. Ruelle, *Statistical Mechanics: Rigorous Results* (World Scientific, 1999).
[36] B. Simon, *The Statistical Mechanics of Lattice Gases*, Vol. I, Princeton Series in Physics (Princeton University Press, 1993).