STAT 315: Functions of Random Variables II: MGF Method

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Moment generating functions $m_X(t) = E[e^t X]$

- Binomial RV $\quad m(t) = \left((1-p) + pe^t\right)^n$
- Geometric RV $\quad m(t) = \dfrac{pe^t}{1-(1-p)e^t}$
- Poisson RV $\quad m(t) = e^{\lambda(e^t - 1)}$
- Exponential RV $\quad m(t) = \dfrac{1}{1-\beta t}$
- Normal RV $\quad m(t) = e^{\mu t + \frac{\sigma^2}{2} t^2}$
- Gamma RV $\quad m(t) = \dfrac{1}{(1-\beta t)^\alpha}$

Theorem (hard to prove)

If $m_X(t) = m_Y(t)$ then X and Y have the same PDF.

Example: MGF of the gamma random variable with parameters α and β.

$$PDF \quad f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} \qquad \Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

$$\begin{aligned} MGF \quad m(y) &= E[e^{tY}] = \int_0^\infty e^{ty} \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)} = \int_0^\infty \frac{y^{\alpha-1} e^{-y(\frac{1}{\beta}-t)}}{\beta^\alpha \Gamma(\alpha)} \\ &= \frac{(\frac{1}{\beta}-t)^{-\alpha}}{\beta^\alpha} \underbrace{\int_0^\infty \frac{y^{\alpha-1} e^{-\frac{y}{(\frac{1}{\beta}-t)^{-1}}}}{(\frac{1}{\beta}-t)^{-\alpha} \Gamma(\alpha)}}_{=1} \\ &= \frac{(\frac{1}{\beta}-t)^{-\alpha}}{\beta^\alpha} = (1-\beta t)^{-\alpha} \end{aligned}$$

Normal and χ^2 (again..)

Suppose Z is standard normal so the PDF is $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ and consider $Y = Z^2$.

$$
\begin{aligned}
E[e^{tY}] &= E[e^{tZ^2}] = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}(1-2t)} dz \\
&= (1-2t)^{-1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}(1-2t)^{-1/2}} e^{-\frac{z^2}{2(1-2t)^{-1}}} dz \\
&= (1-2t)^{-1/2} \qquad \text{since } \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} dz = 1
\end{aligned}
$$

So $Y = Z^2$ has the MGF of a Gamma random variable with $\alpha = 1/2$ and $\beta = 2$. This is also called a χ^2 random variable.

Properties of MGF

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- If $Y = aX + b$ then
$$m_Y(t) = e^{bt} m_X(at)$$

- If Y_1 and Y_2 are independent RV then
$$m_{Y_1+Y_2}(t) = m_{Y_1}(t) m_{Y_2}(t)$$

$$\begin{aligned} m_Y(t) &= E[e^{t(aX+b)}] = E[e^{taX}e^{tb}] = e^{tb}E[e^{taX}] = e^{tb}m_X(at) \\ m_{Y_1+Y_2}(t) &= E[e^{t(Y_1+Y_2)}] = E[e^{tY_1}e^{tY_2}] \underbrace{=}_{\text{indep.}} E[e^{tY_1}]E[e^{tY_2}] \\ &= m_{Y_1}(t)m_{Y_2}(t) \end{aligned}$$

Exponential and Gamma

Example 1: Suppose X is a Gamma RV with parameters α and β. What is $Y = aX$?

Answer: $m_Y(t) = m_{aX}(t) = E[e^{taX}] = m_X(at) = (1 - a\beta t)^{-\alpha}$ so $Y = aX$ is gamma with parameters α and $a\beta$

Example 2: Suppose Y_1 and Y_2 are independent and Gamma random variable with parameters α_1 and β and α_2 and β respectively. What is $Y_1 + Y_2$?

Answer:
$m_{Y_1+Y_2}(t) = m_{Y_1}(t)m_{Y_2} = (1-\beta t)^{-\alpha_1}(1-\beta t)^{-\alpha_2} = (1-\beta t)^{-(\alpha_1+\alpha_2)}$ so $Y_1 + Y_2$ is Gamma with parameters $\alpha_1 + \alpha_2$.

Example 3: Suppose $Y_1, Y_2, \cdots Y_n$ are independent exponential random variable with parameters β. Then the sample average $\frac{Y_1+\cdots+Y_n}{n}$ is a Gamma random variable with parameters n and $\frac{\beta}{n}$. (Combine example 1 and example 2). Mean is $n\frac{\beta}{n} = \beta$ and variance is $n\left(\frac{\beta}{n}\right)^2 = \frac{\beta^2}{n}$

Normal random variables

We proved earlier than the MGF of a normal RV with mean μ and variance σ^2 is $m(t) = e^{\mu t + \frac{\sigma^2}{2} t^2}$.

Example 1: If Y_1 and Y_2 are independent and are normal with mean μ_1 and μ_2 and variance σ_1 and σ_2 then

$$Z = a_1 Y_1 + a_2 Y_2 \text{ is normal with} \begin{cases} \text{mean } a_1\mu_1 + a_2\mu_2 \\ \text{variance } a_1^2\sigma_1^2 + a_2^2\sigma_2^2 \end{cases}$$

Indeed we have

$$\begin{aligned} m_{a_1 Y_1 + a_2 Y_2}(t) &\underbrace{=}_{\text{by indep.}} m_{a_1 Y_1}(t) m_{a_2 Y_2}(t) = m_{Y_1}(a_1 t) m_{Y_2}(a_2 t) \\ &= e^{a_1\mu_1 t + \frac{\sigma_1^2}{2} a_1^2 t^2} e^{a_2\mu_2 t + \frac{\sigma_2^2}{2} a_2^2 t^2} \\ &= e^{(a_1\mu_1 + a_2\mu_2)t + \frac{a_1^2\sigma_1^2 + a_2^2\sigma_2^2}{2} t^2} \end{aligned}$$

Sum of binomial RV or Poisson RV

- $p =$ probability of success in any trial.
- $X_1 =$ number of success in n_1 independent trials.
- $X_2 =$ number of success in n_2 independent trials.

If X_1 and X_2 are independent then $X_1 + X_2$ is the number of success in $n_1 + n_2$ independent trials and so should be binomial

$$\begin{aligned} m_{X_1+X_2}(t) &= m_{X_1}(t) m_{X_2}(t) \\ &= ((1-p) + pe^t)^{n_1} \, ((1-p) + pe^t)^{n_2} \\ &= ((1-p) + pe^t)^{n_1+n_2} \end{aligned}$$

$X_1 + X_2$ is binomial with parameters $n_1 + n_2$ and p

Remark: This works in the same way for sum of independent Poisson RV.

Normal random variables and χ^2 again

Suppose $Z_1, Z_2, \cdots, Z_n$ are independent standard normal then

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

is a sum of n independent gamma RV with parameters $\alpha = \frac{1}{2}$ and $\beta = 2$ and thus

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \text{ is gamma with } \alpha = \frac{n}{2} \text{ and } \beta = 2$$

This is also called a χ^2 RV with n degrees of freedom.