STAT 315: Moment generating functions

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Moment generating functions of a random variable X

So far we have seen two ways to describe a RV

- The PDF $p(x)$ (X discrete) or $f(x)$ (X continuous).
- The CDF $F(X) = P(X \leq x)$.

These are useful to compute

$$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$$

and so on.

We introduce another way, more indirect way, to describe a RV X, the moment generating function of X $m(t)$.. This is a function of a real variable t given by

$$t \mapsto m(t) = E[e^{tX}]$$

The MGF =moment generating function

Moment generating functions

For a random variable the moment generating function $m(t)$ (abbreviated MGF is given by

$$m(t) = E[e^{tY}] \qquad = \sum_y e^{ty} p(y) \quad \text{discrete random variable}$$

$$m(t) = E[e^{tY}] \quad = \int_{-\infty}^{\infty} e^{ty} f(y) dy \quad \text{continuous random variable}$$

Example: Consider a Bernoulli RV Y (that is a binomial random variables with parameters $n = 1$ and p and PDF $p(0) = 1 - p$ and $p(1) = p$. Then the MGF is

$$m(t) = E[e^{tY}] = \sum_x e^{ty} p(y) = p(0)e^0 + p(1)e^t = (1 - p) + pe^t$$

Uniqueness of MGF

The usefulness of MGF comes from the following theorem (which is not easy to prove at all!)

Theorem (Uniqueness)

The moment generating function $m(t)$ determines a random variable uniquely.
More precisely suppose X and Y have moment generating function $m_X(t)$ and $m_Y(t)$ which are finite for t around 0. Then

$$m_X(t) = m_Y(t) \quad \textit{implies} \quad X = Y.$$

We will use that in Section 6!

MGF for binomial RV

- The Binomial Random variable has pdf $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ and so the MGF is

$$m(t) = E[e^{tX}] = \sum_k e^{tk} p(k) = \sum_k e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_k \binom{n}{k} (e^t p)^k (1-p)^{n-k} = (pe^t + (1-p))^n$$

using the binomial theorem $(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$

- The previous theorem tells the converse: if $m_X(t) = (.2e^t + .8)^5$ then X is binomial with parameter $n = 5$ and $p = .2$ and so for example $P(X = 3) = \binom{5}{3}(0.2)^3.(0.8)^3$

Moments from MGF

Moments

If Y is a random variable with MGF $m(t)$ then its moments are

$$E[Y^k] = m^{(k)}(0) \qquad k^{th} \text{ derivative evaluated at } 0$$

Proof: For continuous random variables

$$m'(t) = \frac{d}{dt}\int e^{ty} f(y)dy = \int \frac{d}{dt} e^{ty} f(y)dx = \int y e^{ty} f(y)dy = E[Ye^{tY}]$$

and so $m'(0) = \int y f(y)dy = E[Y]$. Similarly

$$m^{(k)}(t) = \int \frac{d^k}{dt^k} e^{ty} f(y)dy = \int y^k e^{ty} f(y)dy = E[Y^k e^{tY}]$$

and so $m^{(k)}(0) = E[Y^k]$.

MGF for the Gamma random variables

MGF for Gamma

The MGF for a gamma RV is given by

$$m(t) = E[e^{tY}] = \frac{1}{(1-\beta t)^{\alpha}} = (1-\beta t)^{-\alpha}$$

Proof:

$$m(t) = E[e^{tY}] = \int_0^\infty e^{ty} \frac{y^{a-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} dy = \int_0^\infty \frac{y^{a-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\left(\frac{1}{\beta}-t\right)y} dy$$

$$= \frac{\beta(t)^\alpha}{\beta^\alpha} \underbrace{\int_0^\infty \frac{y^{a-1}}{\beta(t)^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta(t)}} dy}_{=1} \quad \text{with} \frac{1}{\beta(t)} = \frac{1}{\beta} - t = \frac{(1-\beta t)}{\beta}$$

$$= \frac{1}{(1-\beta t)^\alpha}$$

Moments for the Gamma random variable

Since

$$m(t) = (1 - \beta t)^{-\alpha}$$

we find

$$m'(t) = \alpha\beta(1 - \beta t)^{-\alpha-1}$$
$$m''(t) = \alpha(\alpha+1)\beta^2(1 - \beta t)^{-\alpha-2}$$
$$m'''(t) = \alpha(\alpha+1)(\alpha+2)(\beta - t)^{-\alpha-3}$$

We find

$$E[Y] = \alpha\beta, \qquad E[Y^2] = \alpha(\alpha+1)\beta^2$$

and so the variance is

$$V(Y) = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

Linear transformations

If Y is a random variable and a and b are constants.

$$\begin{aligned} \text{Mean:} \quad & E[aY+b] = aE[Y]+b \\ \text{Variance:} \quad & V[aY+b] = a^2 V[Y] \\ \text{MGF:} \quad & m_{aY+b}(t) = E[e^{t(aY+b)}] = e^{bt}E[e^{atY}] = e^{bt}m_Y(at) \end{aligned}$$

Example: Suppose Y is exponential with parameter 2 so the MGF is $m(t) = (1-2t)^{-1}$.
Let $Z = 3Y$ then and so

$$m_Z(t) = E[e^{tZ}] = E[e^{3tY}] = m_Y(3t) = (1-2\times 3t)^{-1} = (1-6t)^{-1}$$

By the uniqueness theorem Z is exponential with parameter 6.

MGF for normal RV, part 1

- **Normalization of the standard normal**
  Use polar coordinate

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dx \\
&= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r dr d\theta \quad \text{polar coordinates} \\
&= 2\pi \int_0^{\infty} e^{-s} ds \quad \text{change of variable } s = r^2/2 \\
&= 2\pi
\end{aligned}
$$

and thus

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

MGF for normal RV, part 2

- MGF for the standard normal Z Use that (complete the square)

$$\frac{x^2}{2} - xt = \left(\frac{x^2}{2} - xt + \frac{t^2}{2}\right) - \frac{t^2}{2} = \frac{(x-t)^2}{2} + \frac{t^2}{2}$$

and so

$$\begin{aligned} m_Z(t) = &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t^2/2} e^{-(x-t)^2/2} dx \\ &= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \qquad y = x - t \\ &= e^{t^2/2} \end{aligned}$$

Thus

$$m_Z(t) = e^{t^2/2}$$

MGF for normal RV, 3

- MGF for the normal Y
  With the change of variable $y = (x-\mu)/\sigma$ and $dy = dx/\sigma$

$$\begin{aligned} m_Y(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{xt} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(\mu+\sigma y)} e^{-y^2/2} dy \\ &= e^{t\mu}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{t\sigma y} e^{-y^2/2} dy \\ &= e^{t\mu} m_Z(t\sigma) \\ &= e^{t\mu+\sigma^2 t^2/2} \end{aligned}$$

- By the uniqueness theorem this also proves that

$$Y \sim N(\mu,\sigma^2) \implies Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$$
$$Z \sim N(0,1) \implies Y = \sigma Z + \mu \sim N(\mu,\sigma^2)$$

Example of MGF

	PDF	CDF	MGF
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	$\cdots$	$(pe^t + (1-p))^n$
Geometric	$(1-p)^{k-1} p$	$1-(1-p)^k$	$\frac{pe^t}{1-(1-p)e^t}$
Poisson	$e^{-\lambda} \frac{\lambda^k}{k!}$	.....	$e^{\lambda(e^t-1)}$
Exponential	$\lambda e^{-\lambda t}$	$1-e^{-\lambda t}$	$\frac{\lambda}{\lambda - t}$
Gamma	$\frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}$	$\cdots$	$\frac{1}{(1-\beta t)^\alpha}$
Normal	$\frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$	....	$e^{\mu t + \sigma^2 t^2/2}$
Uniform	$\frac{1}{\theta_2 - \theta_1}$	$\frac{y-\theta_1}{\theta_1-\theta_1}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_1 - \theta_1)}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$	....	.....

CDF calculator: https://homepage.divms.uiowa.edu/~mbognar/ or google colab