STAT 315: Exponential and Gamma Random Variables

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Exponential Random Variables

Exponential Random Variables

A random variable Y is an exponential random variable with parameters β and we write $Y \sim Exp(\beta)$ if the PDF is

$$f(y) = \frac{1}{\beta} e^{-\frac{y}{\beta}} \qquad \beta > 0 \text{ scale parameter}$$

The CDF is given by

$$F(y) = \int_0^y \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = 1 - e^{-\frac{y}{\beta}}$$

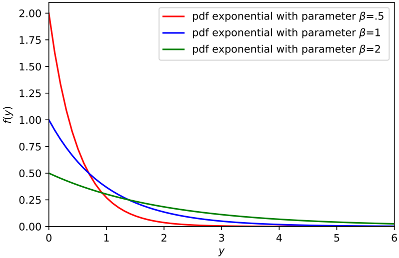
Warning: Very often $\lambda = 1/\beta$ is used a parameter with

$$f(y) = \lambda e^{-\lambda y} \qquad \lambda > 0 \text{ rate parameter}$$

PDF and CDF

PDF: $f(t) = \frac{1}{\beta} e^{-\frac{y}{\beta}}$

CDF: $F(t) = 1 - e^{-\frac{y}{\beta}}$



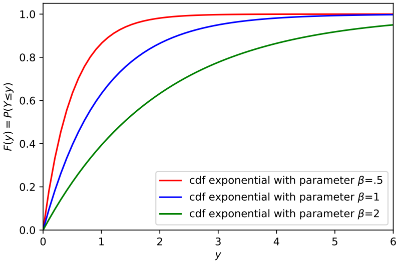


Figure: The Exponential RV with Left:PDF and Right:CDF

Waiting times and the memoryless property

Often Y has the interpretation of a waiting time

- Y is the time between two consecutive earthquake in California
- Y is the time between the emission of two radioactive particles.
- The time it takes to be served at a cash register at a supermarket.
- $\cdots$

Memoryless property

For an exponential random variable Y

$$P(Y \geq t + s | Y \geq t) = P(Y \geq s)$$

"If you have waited for at least 1 hours, the probability you have to wait another 30 minutes is the same as if you just had arrived...."

Mean and Variance of Exponential RV

Moments of exponential RV

For a exponential random variable Y with parameter and β we have

$$E[Y] = \beta \qquad V(Y) = \beta^2$$

Proof: Integration by parts. See the more general computation later.

Gamma Random Variable

Gamma random variables

A gamma random variables Y with parameters α (shape parameter) and β (scale parameter) (we write $Y \sim \Gamma(\alpha, \beta)$) has the pdf

$$f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha} \Gamma(\alpha)} \quad \alpha > 0, \beta > 0$$

Gamma function

The gamma function $\Gamma(\alpha)$ is given by

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

It satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(n) = (n-1)!$ and $\Gamma(1/2) = \sqrt{\pi}$

- Integration by parts with $u = y^\alpha$ and $v' = e^{-y}$

$$\Gamma(\alpha+1) = \int_0^\infty y^\alpha e^{-y} dy = -y^\alpha e^{-y}|_{-\infty}^{\infty} + \alpha \int_0^\infty y^{\alpha-1} e^{-y} dy = \alpha \Gamma(\alpha)$$

- We have $\Gamma(1) = \int_0^\infty e^{-y} dy = 1$ and so

$$\Gamma(n) = (n-1)\Gamma(n-2) = (n-1)(n-2)\Gamma(n-3) = \cdots = (n-1)!$$

- Using the change of variable $y = \frac{x^2}{2}$, $dy = xdx$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy = \int_0^\infty \frac{\sqrt{2}}{x} e^{-\frac{x^2}{2}} x dx = \sqrt{2} \int_0^\infty e^{-\frac{x^2}{2}} dx$$

$$= \sqrt{2} \frac{1}{2} \sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{=1} = \sqrt{\pi}$$

Typical examples where Gamma random variables are used to model non-negative quantities such as for example time until death or time between successive insurance claims).
We can use the two parameters α and β to adjust the mean and variance

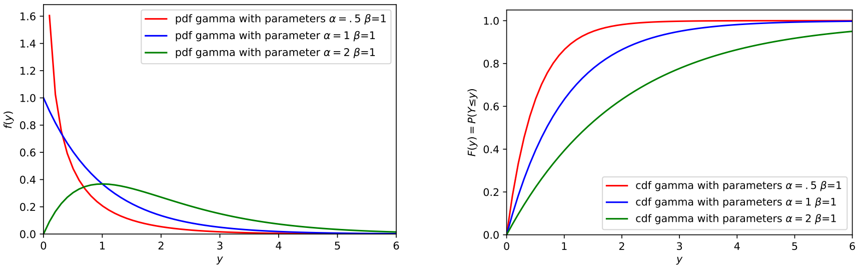


Figure: The gamma RV with Left:PDF and Right:CDF

Mean and Variance of Gamma RV

Moments of Gamma RV

For a gamma random variable Y with parameters α and β we have

$$E[Y] = \alpha\beta \qquad V(Y) = \alpha\beta^2$$

Proof

$$E[Y] = \int_0^\infty y \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)} dy \underbrace{=}_{t=y/\beta} \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty t^\alpha \beta^\alpha e^{-t} \beta dt$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-t} dt = \frac{\beta \Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta\alpha$$

$$E[Y^2] = \int_0^\infty y^2 \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)} dy \underbrace{=}_{t=y/\beta} \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha \Gamma(\alpha)} = \beta^2 \alpha(\alpha+1)$$

χ^2-random variable

χ^2-random variable

A random variable Y is called a χ^2 random variable with k degrees of freedom and we write $Y \sim \chi^2(k)$ if it has the pdf

$$f(y) = \frac{y^{\frac{k}{2}-1} e^{-y/2}}{2^{k/2}\Gamma(k/2)} .$$

- This is a special case of gamma RV with $\beta = 2$ and $\alpha = k/2$ (half-integers).
- There is natural relation between χ^2-random variable and normal random variable (see later). For example if $Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$