STAT 315: Normal Random Variables

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Normal Random Variables

PDF of normal random variables

A continuous random variable Y is normal random variable with parameters $-\infty < \mu < \infty$ and $\sigma > 0$ if it has the density

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \qquad -\infty < y < +\infty$$

We write $Y \sim N(\mu, \sigma^2)$

The CDF

$$F(Y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

has no closed form. Compute it using technology.
See e.g `https://www.webassign.net/tparise/beta/stats/distributionIndex.html`

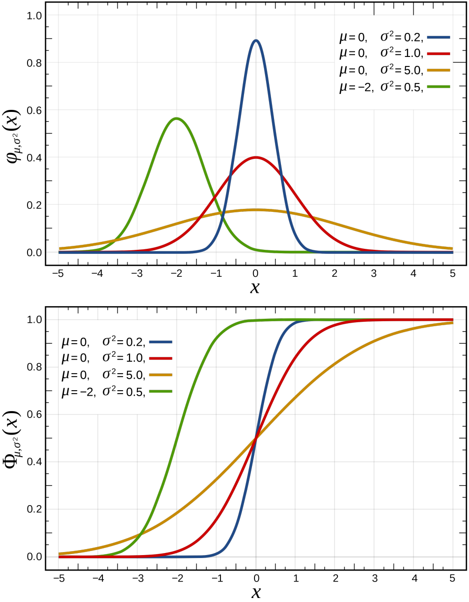
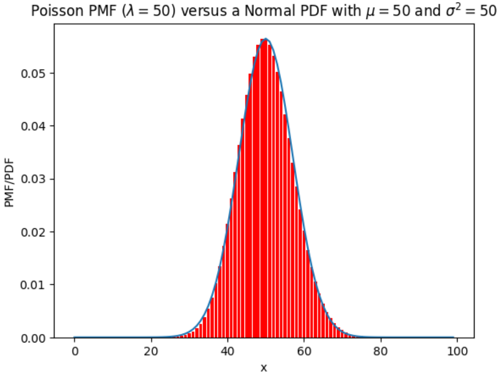


Figure: Top: Pdf, Bottom:Cdf

Why "normal"?

- Normal random variables are , at least in good approximation.
- Many random variables look very close to a normal (under suitable rescaling), e.g



Y Poisson with large λ

- This come from the Central Limit Theorem (Chapter 7)

Standard normal random variable

Standard normal random variable

A normal random variables is called standard normal random variable Z if $\mu = 0$ and $\sigma^2 = 1$. The density is

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \qquad -\infty < y < +\infty$$

Usually we reserve the letter Z is $Z \sim N(0, 1)$

Theorem

If Z is standard normal, $Z \sim N(0, 1)$, then

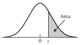
$$Y = \sigma Z + \mu$$

is normal and $Y \sim N(\mu, \sigma^2)$

Normal-table from your book

848 Appendix 3 Tables

Table 4 Normal Curve Areas
Standard normal probability in right-hand tail
(for negative values of *z*, areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).

You can always standardize your RV. If Y is normal with mean μ and variance σ then $Z = \frac{Y-\mu}{\sigma}$ is standard normal.

For example

$$P(a \le Y \le b)$$
$$= P\left(\frac{a-\mu}{\sigma} Z \le \frac{b-\mu}{\sigma}\right)$$

example

- A company that manufactures and bottles apple juice uses a machine that automatically fills 16 ounce bottles. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation of .5 ounce.
  - What is the probability that bottles contains less than 14.5 ounces?
  - Suppose bottle the big enough to contain 17 ounces. What is the probability that the bottles overflow.
  - How big should the bottles be so less than one percent of bottles overflow?

Mean, variance

Mean and Variance of normal RV

If $Y \sim N(\mu, \sigma^2)$ then

$$E[Y] = \mu \qquad V[Y] = \sigma^2$$

Proof: By the previous theorem we can assume $\mu = 0$ and $\sigma^2 = 1$.

$$E[Y] = \int_{-\infty}^{\infty} y \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = 0 \quad \text{(the integrand is odd)}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y \times y \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = -\int_{-\infty}^{\infty} y \frac{d}{dy} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$\underbrace{=}_{IBP} -y \frac{e^{-y^2/2}}{\sqrt{2\pi}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} = 1$$