STAT 315: Mean, variance, and covariance for discrete joint RV

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Expected value of function of joint random variables

If Y_1, Y_2 are joint RV and $g : \mathbb{R}^2 \to \mathbb{R}$ is a function then we can compute the expected value of $g(Y_1, Y_2)$

Expected value

For joint random variables Y_1 and Y_2 and a function $g(Y_1, Y_2)$ we have

$$E[g(Y_1, Y_2)] = \sum_{y_1, y_2} g(y_1, y_2) p(y_1, y_2) \quad \text{discrete RV}$$

Linearity of expected value

Linearity

- For any constant c

$$E[c] = c$$

- For any function $g(Y_1, Y_2)$ and any constant c

$$E[c\, g(Y_1, Y_2)] = c\, E[g(Y_1, Y_2)]$$

- For any functions $g(Y_1, Y_2)$ and $h(Y_1, Y_2)$

$$E[g(Y_1, Y_2) + h(Y_1, Y_2)] = E[g(Y_1, Y_2)] + E[h(Y_1, Y_2)]$$

Same proof as for $f(Y)$!

Independence and products

Independence and products

If Y_1 and Y_2 are independent then for any functions $g(Y_1)$ and $h(Y_2)$

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)]$$

For example independence implies that have

$$E[Y_1 Y_2] = E[Y_1]E[Y_2]$$

Proof: Independence means $p(y_1, y_2) = p(y_1)p(y_2)$ and so

$$\begin{aligned} E[g(Y_1)h(Y_2)] &= \sum_{y_1, y_2} g(y_1)h(y_2)p(y_1)p(y_2) \\ &= \sum_{y_1} g(y_1)p(y_1) \sum_{y_2} h(y_2)p(y_2) \\ &= E[g(Y_1)]E[h(Y_2)] \end{aligned}$$

Covariance

Covariance of Y_1 and Y_2

If Y_1 and Y_2 are random variables with means $\mu_1 = E[Y_1]$ and $\mu_2 = E[Y_2]$ then the covariance of Y_1 and Y_2 is

$$\mathrm{Cov}(Y_1, Y_2) = E\left[(Y_1 - \mu_1)(Y_2 - \mu_2)\right]$$

and the correlation coefficient ρ is

$$\rho = \rho(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_1 \sigma_2}$$

We say that Y_1 and Y_2 are

- positively correlated if $Cov(Y_1, Y_2) > 0$
- negatively correlated if $Cov(Y_1, Y_2) < 0$
- uncorrelated if $Cov(Y_1, Y_2) = 0$

Properties of covariance

1. We have the formula

$$\mathrm{Cov}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$$

2. $\mathrm{Cov}(Y_1, Y_1) = V(Y_1)$ and so $\rho(Y_1, Y_1) = 1$

3. We have Cauchy-Schwartz inequality

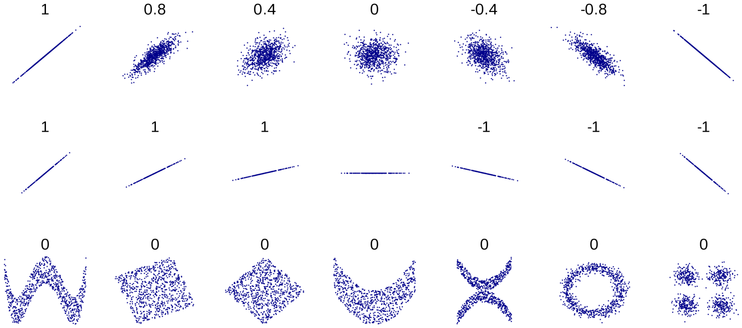
$$|E[Z_1 Z_2]| \le \sqrt{E[Z_1^2]E[Z_2^2]}$$

   and as a consequence the correlation coefficient satisfies

$$-1 \le \rho \le 1$$

4. If Y_1 and Y_2 are independent then $\mathrm{Cov}(Y_1, Y_2) = 0$ and so Y_1 and Y_2 are uncorrelated.
   But the converse is not always true

Example of correlation coefficients



Correlation capture the linear dependence between RV (but not non-linear dependences) (third row)

The correlation reflects the noisiness and direction of a linear relationship (top row), but not the slope of that relationship (second row)

Image taken from Wikipedia

Linear combinations of random variables

For random variables Y_1, Y_2 and Z_1, Z_2 and constants a_1, a_2 and b_1, b_2.

- Expected Value

$$E[a_1 Y_1 + a_2 Y_2] = a_1 E[Y_1] + a_2 E[Y_2]$$

- Variance

$$V(a_1 Y_1 + a_2 Y_2) = a_1^2 V(Y_1) + a_2^2 V(Y_2) + 2 a_1 a_2 \mathrm{Cov}(Y_1, Y_2)$$

- Covariance

$$Cov(a_1 Y_1 + a_2 Y_2, b_1 Z_1 + b_2 Z_2) = a_1 b_1 \mathrm{Cov}(Y_1, Z_1) + \\ + a_1 b_2 \mathrm{Cov}(Y_1, Z_2) + a_2 b_1 \mathrm{Cov}(Y_2, Z_1) + a_2 b_2 \mathrm{Cov}(Y_2, Z_2)$$

Mean and Variance of sample averages

Empirical or sample average

Suppose $Y_1, Y_2, \cdots Y_n$ are independent random variables with

$$E[Y_i] = \mu \qquad V(Y_1) = \sigma^2$$

Then

$$E\left[\frac{Y_1 + Y_2 + \cdots Y_n}{n}\right] = \mu$$

and

$$V\left(\frac{Y_1 + Y_2 + \cdots Y_n}{n}\right) = \frac{\sigma^2}{n}$$

Very important!!