STAT 315-04: Joint discrete random variables

Luc Rey-Bellet

University of Massachusetts Amherst

luc@math.umass.edu

March 6, 2025

Bivariate or joint random variables

Suppose we perform a random experiment. A random variable Y is obtained by performing ONE measurement on the experiment. Joint (or bivariate) random variables Y_1 and Y_2 are obtained by performing TWO simultaneous measurements on the experiment.

Examples:

- We measure the height and weight of some individual in a population.
- We roll three dice and are interested in the number of 1 rolled and the total sum of the dice.

•

• Sampling: We repeat an experiment *n* times and record the results Y_1, Y_2, \dots, Y_n of the experiments (the most important example!)

We need to describe the probability distribution of Y_1 and Y_2 together! This is called the joint (or bivariate) PDF $p(y_1, y_2)$ (discrete)

Joint PDF of discrete RVs.

Joint (or bivariate) PDF for random variables

The joint discrete RV (Y_1, Y_2) have joint a PDF $p(y_1, y_2)$ if

$$P(Y_1 = y_1, Y_2 = y_2) = p(y_1, y_2)$$

with
$$0 \le p(y_1, y_2) \le 1$$
 and $\sum_{y_1, y_2} p(y_1, y_2) = 1$

- Bivariate (=statistics) vs joint (=probability)
- Generalize to more than 2 RV, multivariate RVs.

Joint PDF as table

- 15% of families have 0 children, 20% of families have 1 children, 35% of families have 2 children, 30% of families have 3 children
- boys and girls are equally likely.
- Pick a family at random and let B = number of boys and G = number of girls

		В					
		0	1	2	3		
	0	.15	.1	.0875	.0375		
G	1	.1	.175	.1125			
	2	.0875	.1125				
	3	.0375					

Compute all the entries!

Marginal PDF

We can recover the PDF of Y_1 and Y_2 from the joint PDF. This corresponding to summing rows and columns

Marginal PDF

If the joint discrete RV (Y_1, Y_2) has PDF $p(y_1, y_2)$ then the marginal PDFs of Y_1 and Y_2 are given by

$$p(y_1) = \sum_{y_2} p(y_1, y_2)$$
 $p(y_2) = \sum_{y_1} p(y_1, y_2)$

			В			
		0	1	2	3	
-	0	.15	.1	.0875	.0375	.375
G	1	.1	.175	.1125		.3875
	2	.0875	.1125			.2
	3	.0375				.0375
		.375	.3875	.2	.0375	

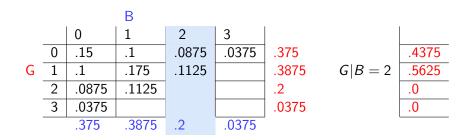
Conditional PDF

Conditional PDF

The conditional PDFs of Y_1 given $Y_2 = y_2$ is given by

$$P(Y_1 = y_1 | Y_2 = y_2) = p(y_1 | y_2) = \frac{p(y_1, y_2)}{p(y_2)}$$

We write $Y_1 | Y_2 = y_2$ for the RV Y_1 conditioned on $Y_2 = y_2$



Independence

• Recall that the events A and B are independent if

P(A|B) = P(A) or P(B|A) = P(B) or $P(A \cap B) = P(A)P(B)$

 Two RV Y₁ and Y₂ are independent if knowing the value for 1 RV one does not influence the probabilities of the other one.

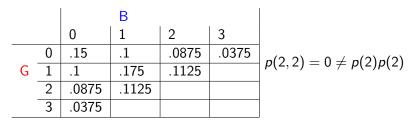
Independence of random variables

The discrete random variables Y_1 and Y_2 are independent if

 $p(y_1|y_2) = p(y_1)$ or $p(y_2|y_1) = p(y_2)$ or $p(y_1, y_2) = p(y_1)p(y_2)$

Example

B and G are not independent (clearly!)



More example

- Two customers enters a store which has 3 cash registers. They both pick a registers at random to check out.
 - Find the joint distribution of register 1 and 2 denoted by (Y_1, Y_2) .
 - ▶ Find the marginal distribution of Y₁ and Y₂?
 - ▶ Are *Y*₁ and *Y*₂ independent?
 - Find the conditional distribution of Y_2 given $Y_1 = 0, 1, 2$?
- Toss three coins successively and consider the random variables *Y*₁=number of Heads obtained

Y = 2 is a side be and takes the value 1 if the first head is on the first coin, value 2 if the first head on the second coin and 3 if the first head on the second coin. If there is no head you lose 1.

- Find the joint distribution of Y_1 and Y_2 .
- Marginal distributions of Y₁ and Y₂?