STAT 315: Poisson Random Variables

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Poisson Random Variable

The Poisson Distribution

A random variable Y is said to have a Poisson probability distribution with parameter $\lambda > 0$ if

- Y takes values $0, 1, 2, 3, \cdots$
- The pdf is

$$p(n) = P(Y = n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

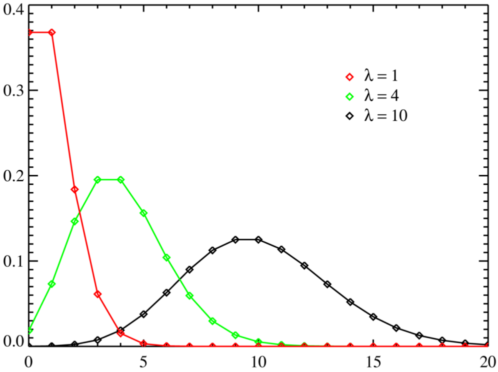
Recall from calculus the series, valid for any x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Thus

$$\sum_{n=0}^{\infty} p(n) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1 \quad \text{as required}$$

Shape of the pdf of a Poisson RV



$$\frac{p(n)}{p(n-1)} = \frac{\frac{\lambda^n}{n!}e^{-\lambda}}{\frac{\lambda^{n-1}}{(n-1)!}e^{-\lambda}}$$

$$= \frac{\lambda}{n} \geq 1$$

$$\text{iff } n \leq \lambda$$

Maximum at $n \approx \lambda$

Mean and Variance of Poisson RV

Mean and Variance

If Y is a Poisson random variable with parameter λ then

$$E[Y] = \lambda \qquad V[Y] = \lambda$$

Proof: $E[Y] = \sum_{n=0}^{\infty} nP(Y = n) = \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} = \sum_{n=1}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda}$ and so

$$E[Y] = e^{-\lambda} \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \underbrace{=}_{k=n-1} \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Similarly

$$E[Y(Y-1)] = \sum_{n=0}^{\infty} n(n-1) \frac{\lambda^n}{n!} e^{-\lambda} = \lambda^2 e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!} = \lambda^2$$

and so $V[Y] = E[Y(Y-1)] + E[Y] - E[Y]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

The Poisson RV is ubiquitous

The Poisson RV describe many different random experiments:

- The number of typos in a slide of your STAT 515 class.
- The number of earthquake hitting Los Angeles county in a given year.
- The number of customer entering a Dunkin Donuts store every minuted during morning rush hour
- The number of rain drops hitting a sq inch in a second
- And many more...

All of them describe some sort of counting processes, counting a number of events for example during a time interval

$$\lambda = \text{rate at which events occur } = E[Y]$$

Why so ubiquitous?

Examples

- The number of typos in any given slide is a Poisson random variables with parameter $1/2$.
  - Probability than one typo occur in one slide
  - Probability more than two typos?
  - Expected number of typos in one page
  - Expected number of typos in 5 pages
  - Probability than three pages are typo-free
  - Probability than two our three pages are typos free?

The law of small numbers or Poisson approximation to binomial

Suppose Y is binomial with parameter n very large and p very small. Mathematically take $n \to \infty$ and $np \to \lambda$ for example take $p = \frac{\lambda}{n}$.

$$
\begin{aligned}
p(k) &= \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \times \frac{n(n-1)\cdots(n-k+1)}{n^k} \times \left(1 - \frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda} \text{ as } n \to \infty} \times 1 \underbrace{\left(1 - \frac{1}{n}\right)}_{\to 1 \text{ as } n \to \infty} \cdots \underbrace{\left(1 - \frac{k-1}{n}\right)}_{\to 1 \text{ as } n \to \infty} \times \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{\to 1 \text{ as } n \to \infty} \\
&\to \frac{\lambda^k}{k!} e^{-\lambda} \text{ as } n \to \infty
\end{aligned}
$$

Recall from Calculus (use L'Hospital rule) that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

Poisson approximation in picture

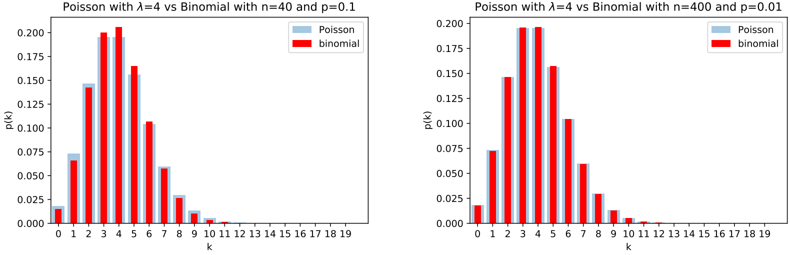


Figure: Left: $n = 40,\ p = .1,\ np = \lambda = 4$. Right: $n = 400, p = .01,\ np = \lambda = 4$

$$\text{Poisson } (\lambda = 4) \quad P(Y = 5) = e^{-4}\frac{4^5}{5!} = 0.1562$$

$$\text{Binomial } (n = 40, p = .1) \quad P(X = 5) = \binom{40}{5}(.1)^5(.9)^{35} = 0.1647$$

$$\text{Binomial } (n = 400, p = .01) \quad P(X = 5) = \binom{400}{5}(.01)^5(.99)^{35} = 0.1570$$

Poisson approximation

If we have a large numbers of trials and the probability of success is small then a Poisson distribution will be a very good description (even if the trial are not fully independent.

Birthday problem (again):

• N people in the room $\rightarrow \binom{N}{2}$ pairs of people. So $n = \binom{N}{2}$ (This is large: e.g. $N = 42, \binom{N}{2} = 861$)

• Probability that a pair of people share a birthday is $p = \frac{1}{365}$ (small!)

• Poisson approximation

$$Y = \text{ number of pair people sharing the same birthday}$$

is approximately Poisson with $\lambda = np = \binom{N}{2}\frac{1}{365} = \frac{N(N-1)}{730}$

• Then we find

$$P(\text{no people have the same birthday}) \approx P(Y = 0) = e^{-\lambda} = e^{-\frac{N(N-1)}{730}}$$