

STAT 315: Hypergeometric Random Variables

Luc Rey-Bellet

University of Massachusetts Amherst

luc@math.umass.edu

March 4, 2025

Hypergeometric Random Variable

Hypergeometric Random Variable

- $N =$ size of the population
- Population members come in two kinds: r are of type I and $N - r$ are of type II .
- Take a sample of size n in the population. (n must be less than or equal to N).
- The hypergeometric random variable Y gives the number of samples which are of type I .
- The pdf of Y is

$$p(k) = P(Y = k) = \frac{\binom{r}{k} \binom{N-r}{n-k}}{\binom{N}{n}}$$

The number of possible samples is $\binom{N}{n}$. If $Y = k$ then there are $\binom{r}{k}$ ways for the sample to give k type I and $\binom{N-r}{n-k}$ ways to $n - k$ type II .

Sampling with or without replacement

Mean of the hypergeometric RV

If X is hypergeometric with parameters N =total population size, r =number of type I, n =sample size. Then

$$E[X] = \frac{nr}{N}$$

Hypergeometric RV = **sampling without replacement**: you pick a sample of size n out a population of N .

Sample of size n **with replacement** out of a population of size N with r elements of type I gives a **binomial with probability of success $p = \frac{r}{N}$** .

$$P(X = k) = \underbrace{\frac{\binom{r}{k} \binom{N-r}{n-k}}{\binom{N}{n}}}_{\text{without replacement}} \quad \text{vs} \quad \underbrace{\binom{n}{k} \left(\frac{r}{N}\right)^k \left(\frac{N-r}{N}\right)^{n-k}}_{\text{with replacement}}$$

Sampling with or without replacement in picture

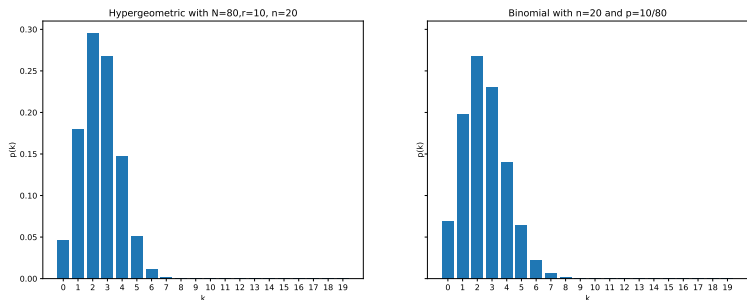


Figure: Left: without replacement (hypergeometric). Right: with replacement (binomial). If N is very large one expects that sampling with or without replacement will look quite similar.

Keno game (often played on video in bars, new draw every few minutes)

Game ticket

KENO
Underline Your Number Selections Like This: 5

1 Quantity of Numbers (spots) You Want Per Game.
1 2 3 4 5 6 7 8 9
10 11 12

2 Amount Bet Per Game.
1 2 5 10 20

3 Number of Games Played.
1 2 3 5 10 20 30

4 Pick Your Own Numbers by Marking the Boxes You Want or Mark the Quic Pic Box.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80

5 Do you want to play KENO BONUS? (Mark Box)
Playing KENO BONUS can increase KENO prizes by as much as 10 times.
If you play KENO BONUS the ticket amount amount will double.

Quic Pic

INSERT THIS END

Play KENO - It's Exciting Action.

- The player select r numbers (for example $r = 10$).
- 20 numbers are drawn from 80 numbers
- Payout if you k of your 10 numbers match.

How to Win

Number of Spots

10 Spot Game

Win Type	Prize*	Probability
Match 10	\$100,000	1:8,911,711.18
Match 9	\$10,000	1:163,381.37
Match 8	\$500	1:7,384.47
Match 7	\$80	1:620.68
Match 6	\$20	1:87.11
Match 5	\$2	1:19.44
Match 0	\$2	1:21.84
Overall odds are 1 in 9.05		

*See Game Rules for details on prize limitations. All prizes above based on a \$1 wager.

Keno

For a 10 spot game, we have

$N = 80$, $r = 10$ (=the numbers you have chosen)

$n = 20$ sample size, since 20 numbers are sampled out of 80

$$p(k) = P(\text{match of } k) = \frac{\binom{10}{k} \binom{70}{20-k}}{\binom{80}{20}}$$

Expected payout for a bet of \$1.

$$\begin{aligned} E[P] = & 2 \times p(0) + 2 \times p(5) + 20 \times p(6) + 80 \times p(7) + 500 \times p(8) \\ & + 10,000 \times p(9) + 100,000 \times p(10) = 0.6930 \end{aligned}$$

Ouch.....