STAT 315: Geometric and Negative Binomial Random Variables

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Geometric random variable

Geometric distribution

- An experiment consist of successive identical and independent trials
- Each trial results in success S with probability p or failure F with probability $1-p$.
- The geometric random variable N is defined by

$$N = \text{ number of trial until the first success}$$

- The pdf of a geometric random variable is

$$p(k) = P(N=k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \cdots$$

We have $N = k$ if the first success occurs on the k^{th} trial i.e. we have the sequence of trials

$$\underbrace{F, F, \cdots, F,}_{k-1} S \text{ which has probability } (1-p)^{k-1}p$$

Math reminder: Geometric series

To compute the expected value and the variance of a geometric random variables we will need the following series (see you calculus class).

Geometric series

Provided $|x| < 1$ we have the following infinite series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 \cdots = \frac{1}{1-x}$$

$$\sum_{k=1}^{\infty} kx^{k-1} = x + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

$$\sum_{k=2}^{\infty} k(k-1)x^{k-2} = 1 + 2 + 6x + 12x^2 + \cdots = \frac{2}{(1-x)^3}$$

Survival probability and the memoryless property

If N is a geometric RV with probability p then think as p to be the probability to "die" in a given time period. Then $P(N > n) =$ Probability to "survive" at least n periods of time

$$\begin{aligned} P(N > n) &= P(N = n+1) + P(N = n+2) + \cdots \\ &= (1-p)^n p + (1-p)^{n+1} p + (1-p)^{n+2} p + \cdots \\ &= p(1-p)^n \left[1 + (1-p) + (1-p)^2 + \cdots\right] \\ &= p(1-p)^n \frac{1}{1-(1-p)} = (1-p)^n \end{aligned}$$

and so

$$P(N > n+k | N > n) = \frac{P(N > n+k)}{P(N > n)} = \frac{(1-p)^{n+k}}{(1-p)^n} = (1-p)^k$$

The probability to survive k more periods if you have survived n periods is $(1-p)^k$ (independent of n!) (Memoryless property, you are reborn every day!).

Expected value and variance of geometric random variables

Mean and Variance of a Geometric Random Variable

$$E[N] = \mu = \frac{1}{p} \qquad V[N] = \sigma^2 = \frac{1-p}{p^2}$$

Proof: The PDF is $p(k) = (1-p)^{k-1}p$ for $k = 1, 2, 3, \cdots$
So

$$E[N] \;=\; \sum_{k=1}^{\infty} kp(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p\frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

So the mean is $E[N] = \frac{1}{p}$. This is intuitive. If the probability of success is $1/10$ then one average it takes 10 trials to succeed!

For the variance instead of $E[N^2]$ compute again $E[N(N-1)]$.

$$
\begin{aligned}
E[N(N-1)] &= \sum_{k=1}^{\infty} k(k-1)p(k) = p(1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} \\
&= p(1-p)\frac{2}{(1-(1-p))^3} = p(1-p)\frac{2}{p^3} = \frac{2(1-p)}{p^2}
\end{aligned}
$$

So

$$
\begin{aligned}
V[N] &= E[N^2] - E[N]^2 \\
&= E[N(N-1)] + E[N] - E[N]^2 \\
&= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{2(1-p)}{p^2} + \frac{p-1}{p^2} = \frac{(1-p)}{p^2}
\end{aligned}
$$

Example

- An oil prospector is looking for oil in Oklahoma It costs \$10,00 to drill a well and the probability to find oil in each well is $\frac{1}{8}$.
  - If the prospector has $0,000$? What is the probability the prospector will go bankrupt?
  - What is the expected amount of money you need to get one functioning well. What about 5 functioning wells?
-

Example: Bitcoin mining

• Fun (or Scary) fact: Bitcoin mining is using as much electricity as the entire country of Finland (circa 5 million people). Why?

• Bitcoin is a cryptocurrency where no central authority keeps track of transactions. To record transactions there is a lottery and whoever wins the lottery can record a block of transactions which are added to "the block chain"). He is rewarded by earning 3.125 bitcoins (1 BTC = 85,565.98 on 2/27/2025). The reward is halved every four years. `https://bitinfocharts.com/bitcoin/`

• The winner of the lottery must produce a hash value for the hash function SHA256 which is less than some value, that is a 256bits starting with k 0's (this has probability $1/2^k$). This is done by farms of specialized computers ("the miners").

• The number of 0, k, needed is adjusted so that it takes about 10 minutes to get a winner. This this depends on the number of players! If the entire sets of miners can do 2^k SHA256 computations every 10 minutes then it will take on average 10 minutes to produce a winner since $E[N] = \frac{1}{p}$.

Negative binomial random variables

We count the number of trials until $r = 2, 3, 4, 5, \cdots$ successes.

Negative Binomial Random Variable

- An experiment consist of successive identical and independent trials
- Each trial results in success S with probability p or failure F with probability $1 - p$.
- A negative binomial random variable N with parameter r, p is

  $N =$ number of trial until r successes. $N = r, r+1, r+2, \cdots$
- The pdf of a negative binomial variable is

$$p(k) = P(N = k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r \qquad k = r, r, +1, r+2$$

The last, k^{th}, trial is a success (the r^{th} one) and the other $r - 1$ successes can be any of the other $k - 1$ trials, hence the binomial $\binom{r-1}{r-1}$

Example: NBA playoffs series

- The Boston Celtics are playing the Denver Nuggets
- The team which wins 4 games first wins the playoff series.
- Assume that each game is won independently by the Celtics with probability p=.45
- The probability the Boston Celtics wins the series is described by a negative binomial RV Y with $r = 4$ since the Celtics must accumulate 4 wins before the 7 games.

$$P(\text{ Celtics wins}) = P(Y = 4) + P(Y = 5) + P(Y = 6) + P(Y = 7)$$
$$= p^4 + \binom{4}{0} p^4(1-p) + \binom{5}{2} p^4(1-p)^2 + \binom{6}{3} p^4(1-p)^3$$
$$= 0.3917...$$

Geometric and negative binomials PDF

The geometric RV is a negative binomial RV with $r = 1$.

$E[N] = \frac{1}{p}$, $V[N] = \frac{(1-p)}{p^2}$

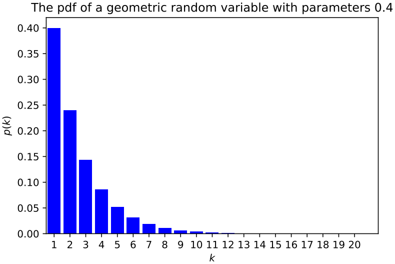

Figure: Geometric pdf
$p(k) = (1-p)^{k-1}p$

$E[N] = \frac{r}{p}$, $V[N] = \frac{r(1-p)}{p^2}$

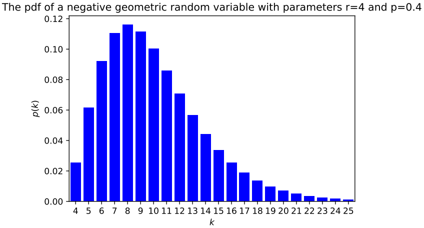

Figure: Negative binomial pdf
$p(k) = \binom{k-1}{r-1}(1-p)^{k-1}p^r$

Remark: we will compute mean and variance of negative binomials later.