STAT 315: Functions of Random Variables and Variance

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Functions of random variables

Function of random variable

If Y is a random variable with pdf $p(y)$ $g : \mathbb{R} \to \mathbb{R}$ is a function then $Z = g(Y)$ is another random variable with pdf

$$P(Z = z) = P(g(Y) = z) = \sum_{y : g(y) = z} p(y)$$

Example: If Y takes values $-1, 0, 1, 2$ with pdf

$$P(Y = -1) = \frac{1}{8}, P(Y = 0) = \frac{3}{8}, P(Y = 1) = \frac{1}{4}, P(Y = 2) = \frac{1}{4}$$

then $Z = X^2$ takes values $0, 1, 4$ with pdf

$$\begin{aligned} P(Z = 0) = P(Y = 0) & = 3/8\,, \\ P(Z = 1) = P(Y = -1) + P(Y = 1) & = 1/8 + 1/4 = 3/8\,, \\ P(Z = 4) = P(Y = 2) & = 1/4\,. \end{aligned}$$

Expected value of $Z = g(Y)$

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The expected value of the random variable $Z = g(Y)$ is given by

$$E\left[g(y)\right] = \sum_y g(y)p(y)$$

Proof:

$$E[g(Y)] = E[Z] = \sum_z zp(z) = \sum_z z \sum_{y:g(y)=z} p(y) = \sum_y g(y)p(y)$$

Example cont'd: Two ways to compute $E[Y^2]$!

- $E[Y^2] = 0 \times \frac{3}{8} + 1 \times \frac{3}{8} + 4 \times \frac{1}{4} = \frac{3}{2}$
- $E[Y^2] = (-1)^2 \times \frac{1}{8} + 0^2 \times \frac{3}{8} + 1^2 \times \frac{1}{4} + 2^2 \frac{1}{4} = \frac{3}{2}$

Examples

- An insurance company insures against a certain risk and will pay \$1000 if the accident occurs. It is know that 3% of the policy holders will incur the accident during any single year. The fixed administrative cost of issuing the policy is \$20. What premium C should the insurance company charge to make a profit of \$50 per customer.

Properties of the expectation: Linearity

Properties of the expectation

If Y is a random variable with pdf $p(y)$ and c is a real number

- $E[1] = 1$
- $E[cg(Y)] = cE[g(Y)]$
- $E[g_1(Y) + g_2(Y)] = E[g_1(Y)] + E[g_2(y)]$

Proof: We have $E[1] = \sum_y 1p(y) = \sum_y p(y) = 1$. Further

$$E[cg(Y)] = \sum_y cg(y)p(y) = c\sum_y g(y)p(y) = cE[g(Y)]$$

and

$$\begin{aligned} E[g_1(Y) + g_2(Y)] &= \sum_y (g_1(y) + g_2(y))p(y) \\ &= \sum_y g_1(y)p(y) + \sum_y g_2(y)p(y) \\ &= E[g_1(Y)] + E[g_2(Y)] \end{aligned}$$

The variance and standard deviation of a random variable

The variance $V[Y]$

If Y is a random variable with mean $E[Y] = \mu$ then the variance of Y, $V(y)$ is the expected value of $(Y - \mu)^2$:

$$V[Y] = E\left[(Y-\mu)^2\right] = E\left[(Y - E[Y])^2\right]$$

Usually we use the greek letters μ (the greek "m") and σ (greek "s").

$$E[Y] = \mu \quad \text{and} \quad V(Y) = \sigma^2$$

The standard deviation

The standard deviation of Y is given by

$$\sigma = \sqrt{V[Y]} = \sqrt{E\left[(Y-\mu)^2\right]}$$

Formula for the variance

Useful formula to compute the variance

$$V[Y] = E[Y^2] - E[Y]^2$$

Proof:

$$\begin{aligned}
V[Y] &= E[(Y - E[Y])^2] \\
&= E\left[Y^2 - 2E[Y]Y + E[Y]^2\right] \\
&= E[Y^2] + E[\underbrace{-2E[Y]}_{=constant} Y] + E[\underbrace{E[Y]^2}_{=constant}] \\
&= E[Y^2] - 2E[Y]E[Y] + E[Y]^2 \underbrace{E[1]}_{=1} \\
&= E[Y^2] - 2E[Y]^2 + E[Y]^2 \\
&= E[Y^2] - E[Y]^2
\end{aligned}$$

The theorem is very useful to compute the variance.

Example: If Y takes values $-1, 0, 1, 2$ with pdf

$$P(Y=-1)=\frac{1}{8}, P(Y=0)=\frac{3}{8}, P(Y=1)=\frac{1}{4}, P(Y=2)=\frac{1}{4}$$

$$\mu = E[Y] = -\frac{1}{8} + \frac{1}{4} + 2\frac{1}{4} = \frac{5}{8}, \quad E[Y^2] = \frac{1}{8} + \frac{1}{4} + 4\frac{1}{4} = \frac{11}{8}$$

$$\sigma^2 = V(Y) = E[Y^2] - E[Y]^2 = \frac{11}{8} - \frac{25}{64} = \frac{63}{64}$$

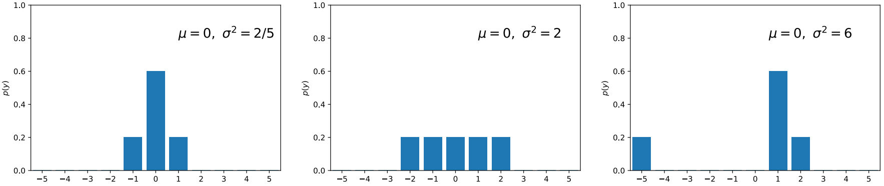
But using the definition we have

$$\begin{aligned}
\sigma^2 =& V(Y) = E[(Y-\mu)^2] \\
&= \left(-1-\frac{5}{8}\right)^2 \frac{1}{8} + \left(0-\frac{5}{8}\right)^2 \frac{3}{8} + \left(1-\frac{5}{8}\right)^2 \frac{1}{4} + \left(2-\frac{5}{8}\right)^2 \frac{1}{4} \\
&= \text{(after mildly painful algebra) } \frac{63}{64}
\end{aligned}$$

Interpretation of the variance

Variance = measure of the spread of the pdf around the mean

- $\mu = E[Y]$=average value of Y.
- $\sigma^2 = V[Y] = E[(Y - E[Y])^2]$ =average squared distance to the mean
- $\sigma = \sqrt{V[Y]}$ = average distance to the mean



Variance=measure of risk

- For a bet of \$1 would rather bet on RED or bet on the number 23? Compute the corresponding variance?
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Linear transformation

Expectation and variance under linear transformations

$$E[aY+b] = aE[Y]+b$$
$$V(aY+b) = a^2V(Y)$$

Proof: For the expectation

$$E[aY+b] = E[aY]+E[b] = aE[Y]+b \quad \text{By linearity of expectation}$$

For the variance use the definition and the formula for the expectation

$$\begin{aligned} V(aY+b) &= E[(aY+b-E[aY+b])^2] = E[(aY+b-aE[Y]+b)^2] \\ &= E[a^2(Y-E[Y])^2] = a^2E[(Y-E[Y])^2] = a^2V(Y) \end{aligned}$$

□

The variance is unchanged under translation $Y \to Y+b$ since only the relative distance to the mean matters.