STAT 315-06: Conditional probability and independence (Section 2.7)

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What is conditional probability?

- **Super important concept! (and super useful too!)**
- Quantify the learning process. Suppose we perform (partial) observations on a random experiment (i.e. observe that some event occurs). What can we learn from this?
- Setup:
  - We are interested in the event A which has probability

$$P(A) \qquad (= \text{"Prior probability"}).$$

  - We observe that the event B has occured.
  - How does this change the probability that A occur?

$$P(A|B) \qquad (= \text{"Posterior probability"}).$$

=Probability that A occurs given that B has occurred.

Motivation for the definition

Example: 282 persons were asked whether they like like Tom B or not?

	New England	Rest of the country	
YES	2	20	22
NO	162	98	260
	164	118	282

- For a randomly chosen individual we have $P(\text{ like Tom B}) = \frac{22}{282}$.
- If we can assert who is from New England then for a randonly chosen New Englander (the sample space is reduced to 164 individuals) we have $P(\text{like Tom B}|\text{New England}) = \frac{P(\text{NE \& likes Tom B})}{P(NE)} = \frac{2}{164}$
- Similarly $P(\text{New England}|\text{like Tom B.}) = \frac{2}{22}$

Conditional probability: mathematical definition

Definition

Conditional Probability The conditional probability of the event A given that an event B has occurred is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

In short we say "the probability of A given B"

Intuition:

- Before observing B the sample space is S and the probability of A is $P(A)$.
- After B has been observed, the sample space shrinks from S to $S \cap B = B$. Now for A to occur the outcome must belong to $A \cap B$ and so $P(A|B) \propto P(A \cap B)$. Dividing by $P(B)$ ensures that $P(A|B)$ is a probability.

Sensitivity and specificity of a test

Suppose you are building a test to detect some disease. The quality of a test is measured two quantities

- The sensitivity (PPA) of a test is the probability to test positive given that you are infected.
- The specificity (PNA) of a test is the probability to test negative if healthy
- The FDA standards (found here) are $PPA > .9$ and $PNA > .95$
- These are conditional probabilities: consider the events

$$I = \{\text{infected}\}, \quad H = \overline{I} = \{\text{healthy}\}$$
$$Pos = \{\text{ test positive}\}, \quad Neg = \overline{Pos} = \{\text{ test negative}\}$$

Then

$$\text{Sensitivity} = P(Pos|I) \qquad \text{Specificity} = P(Neg|H)$$

- More on this later!

Examples

- You have a bag with 4 red balls, 3 blue balls, and 5 green balls. You randomly draw one ball. What is the probability that the ball is blue, given that it is not green?
- A family has 2 children. I know that one of the children is a boy. Given this piece of information find the probability that both children are boys?
- Draw two cards from a standard deck of 52 cards. Consider the following events.

  $$A = \{\text{1st card is an ace}\}, \quad B = \{\text{at least one card is an ace}\},$$
  $$D = \{\text{both cards are aces}\}$$

  Compute $P(D|A)$ and $P(D|B)$.

Properties of conditional probability

- If $A \cap B = \emptyset$ (mutually exclusive) then $P(A|B) = 0$: If B occurs then A cannot occur!.
- If $B \subset A$ then $P(A|B) = 1$: if B has occurred then A occurs for sure. Special cases $P(B|B) = 1$ and $P(S|B) = 1$.
- For fixed B, $P(A|B)$ is a probability.
  1. $P(A|B) \geq 0$, ✓
  2. $P(S|B) = 1$, ✓
  3. If $A_1 \cap A_2 = \emptyset$ then

$$\begin{aligned} P(A_1 \cup A_2|B) &= \frac{P\left((A_1 \cup A_2) \cap B\right)}{P(B)} = \frac{P\left((A_1 \cap B) \cup (A_2 \cap B)\right)}{P(B)} \\ &= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} \\ &= P(A_1|B) + P(A_2|B) \checkmark \end{aligned} \tag{1}$$

Independence

Definition of independence I

The event A is independent of B if the occurrence of B has no influence on the occurence of A, that is

$$P(A|B) = P(A)$$

$$\text{Since } P(A|B) = P(A) \Leftrightarrow \frac{P(A \cap B)}{P(B)} = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$$

Definition of independence II

The events A and B are independent if any of the following holds

$$\begin{aligned} P(A|B) &= P(A) \quad (A \text{ independent of } B) \\ P(B|A) &= P(B) \quad (B \text{ independent of } A) \\ P(A \cap B) &= P(A)P(B) \end{aligned}$$

Examples

- Draw two cards from a deck (with replacement)

$$A = \{\text{first card is a heart}\}, B = \{\text{second card is a spade}\}$$

  Are A and B independent?
  What if you do it without replacement?

- Roll two dice. Consider the events

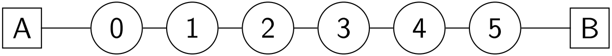
$$A = \{\text{sum is } 6\}, B = \{\text{sum is } 7\}, C = \{\text{1st dice is } 4\}$$

  Are A and C independent? Are B and C independent?

- Show that if A and B are independent then A and $\overline{B}$ are also independent.

Circuits

- All nodes are independent form each other and are open with probability p and closed with probability $1 - p$.
- We are interested to find an open path between two points though a network of nodes.
- **Nodes in series**

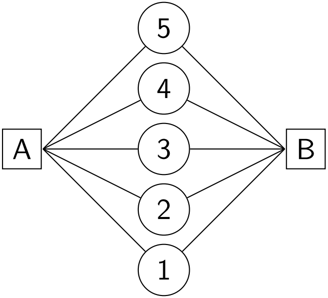


What is the probability to have a path from A to B?

$$
\begin{aligned}
P(\text{Path } A \to B) &= P(\text{all nodes open}) \\
&= P(1 \text{ open} \cap 2 \text{ open} \cap \cdots \cap 5 \text{ open}) \\
&= P(1 \text{ open})P(2 \text{ open}) \cdots P(5 \text{ open}) = p^5
\end{aligned}
$$

Circuits (continued)

- **Nodes in parallel**



$$
\begin{aligned}
P(\text{Path } A \to B) &= P(\text{at least one nodes open}) \\
&= 1 - P(\text{all nodes closed}) \\
&= 1 - P(1 \text{ closed} \cap 2 \text{ closed} \cap \cdots \cap 5 \text{ closed}) \\
&= 1 - P(1 \text{ closed})P(2 \text{ closed}) \cdots P(5 \text{ closed}) \\
&= 1 - (1-p)^5
\end{aligned}
$$

Example

Compute the probabilities that a path from A to B is open for the circuits (all gates are independent and open with probability p)