STAT 315: Counting (Section 2.6)

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Why is counting important?

Many problems can be described by a uniform probability distribution on the sample space S, that is, all elements in the sample space S are equally likely. For each $x \in S$ we have

$$P(\{x\}) = \frac{1}{|S|} \quad \text{where } |S| = \text{cardinality of } S$$

or more generally for any event A

$$P(A) = \frac{|A|}{|S|} = \frac{\#\text{ of elements in } A}{\#\text{ of elements in } S}$$

- One need to compute the cardinality $|S|$ of S.
- This can be a hard problem. Compute the number of n-bits without consecutive 1. Doable...
- Compute the number of $n \times n$ matrices with 0 or 1 entries and without adjacent 1. Super hard...

Sample space size

In many problems the sample space is HUGE!

- There are 42 students enrolled in this class and so there are

$$42! = 1.4 \times 10^{51}$$

  ways to arrange the students in an ordered list. By comparison there are (more or less) 1.33×10^{50} atoms on earth.

- A Hash function (see Wikipedia page), used for example in data storage, bitcoin mining, see later, cryptography, etc.. maps any data of arbitrary length into a sequences of bits of fixed size. For example SHA-256 maps any data into a sequence of 256 bits (there are $2^{256} = 16^{64} = 1.15 \times 10^{77}$).
  A good (cryptographic) hash function is
  - Easy to compute, see online SHA256 calculator
  - Creates a (nearly) uniform distribution on the values of the Hash function (can be used as a random number generator).
  - Practically impossible to invert the function, that it to recover the data from the value of the hash function.

Basic Principle of counting

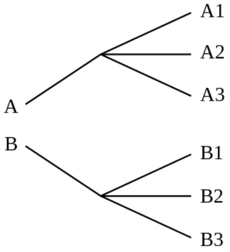
Basic principle of counting

Two experiments are being performed. If experiment 1 has n possible outcome and if for each outcome of experiment 1 there are m possible outcomes for experiments 2 then there are $n \cdot m$ possible outcomes of the two experiments.

This generalizes immediately to r experiments with $n_1, n_2, \cdots, n_r$ possible outcomes at each step. In this case, we have

$$n_1 \cdot n_2 \cdot n_3 \cdots n_r$$

possible outcomes.



Examples

• Connecticut license plates (Serties 2013) consist a number (except 0) followed by four letters (C, I, O, Q, Y, and Z are never used) followed by a numbers.
How many license plates are possible?

Permutations: ordered lists

Permutations and factorials

If you have n objects and you make a list (in a certain order) of these n objects then there are

$$n! = n(n-1)(n-2)\cdots 1$$

choices for the list. Indeed there are n choices for the first one on the list, $n-1$ choices for the second on the list, and so on....

To get a sense of how large $n!$ easy it is useful to know Stirling's formula

Stirling's formula

For large n we have

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

where $a_n \sim b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Binomial coefficients: selecting r objects out of n objects

Question: If we have n items at our disposal, how many groups of r objects can we form?

Example: If we have 10 people there are $10 \cdot 9 \cdot 8 = \frac{10!}{7!}$ ways to select 3 people (in order). But the exact same 3 people will have appear in different list in a different order (in 3! different lists). Thus there are

$$\frac{10 \cdot 9 \cdot 8}{3!} = \frac{10!}{3!7!}$$

ways to select them. More generally we have

Binomial coefficients: selecting r objects out of n objects

- There is $C(n,r) = \binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$ ways to select r objects out of n objects (no order)
- There is $P(n,r) = \frac{n!}{(n-r)!}$ ways to make an ordered list of r objects out of n objects.

Examples

- Professor L. has 20 math books, 10 physics books, and 5 CS books. How many ways can he arrange his books in his bookshelf if he wants to keep the book of the same subject together?
- Among the 42 students 23 are CS-Math majors and 19 from other majors. How many ways can I pick a group of three students among them? What if I don't want all of them to have the same majors?
- How many words can you write using the letters MISSISSIPPI using all letters?

Pascal's triangle

Recursion relation for binomial coeficients

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof Think of a group of n people as $n-1$ people (say the students in the class) plus 1 (the professor).

- $\binom{n}{k}$ = number of ways to select a group of size k in the class.
- $\binom{n-1}{k}$= number of ways to select a group of k consisting only of students (the professor is not included)
- $\binom{n-1}{k-1}$= number of ways to select a group of k which contain the professor. Pick the professor and select $k-1$ students.

QED

Pascal's triangle and earlier versions

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

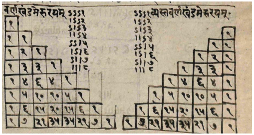


Figure: Meru Prastaara 755 A.D from Wikipedia

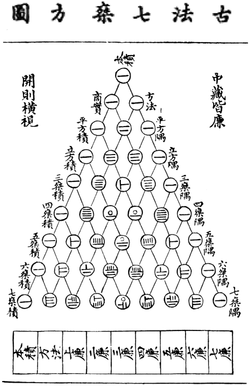


Figure: Yang Hui's triangle (1303) from Wikipedia

Poker hands

POKER HAND RANKINGS
PROBABILITY OF MAKING A SPECIFIC HAND
Royal flush ~0.0001%
Straight flush ~0.0013%
Four-of-a-kind ~0.0240%
Full house ~0.1440%
Flush ~0.1965%
Straight ~0.3924%
Three-of-a-kind ~2.1128%
Two pair ~4.7539%
Pair ~42.2569%
High card ~50.1177%

Examples

- How many poker hands are there?
- Probability of 4 of a kind (e.g. 4 aces of 4 eights)?
- Probability of full house (i.e. 3 of a kind and a pair)?
- Probability of three of kind (i.e. 3 of a kind and a pair)?

Coincidences (The Birthday problem)

- See https://en.wikipedia.org/wiki/Birthday_problem for more info.
- The birthday problem asks for the probability that, in a set of K randomly chosen persons, at least two will share a birthday.
- Assume that (a) every birthday is equally likely (not quite true) and (b) no leap years.
- Sample space $S = \{(d_1, \cdots d_K)\, d_i \in \{1, 2, \cdots, 365\}\}$
- Consider the complementary event

$$A = \{\text{no birthday is shared by 2 or more persons}\}$$

To be in A we we must have $d_i \neq d_j$ for $i \neq j$.

Birthday problem continued

- The sample space has $|S| = 365 \times 365 \times \cdots \times 365 = 365^K$ elements.
- Cardinality of A. The first person, birthday can be anything, but the second must be different from the first (364 possibilities), and the third one distinct from the the first two (363 possibilities), and so on

$$|A| = 365 \times 364 \times 363 \times \cdots \times (365 - k + 1)$$

- Probability of A

$$P(A) = \frac{|A|}{|S|} = \frac{365 \times 365 \times 365 \times \cdots \times 365}{365 \times 364 \times 363 \times \cdots \times 365 - k + 1}$$
$$= \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{k-1}{365}\right)$$

Birthday problem continued

For a room with 23 people $P(A) = .5$. For a room with 42 people we have $P(A) = .09$.

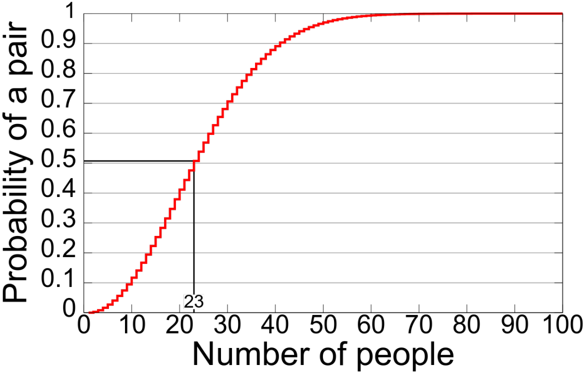


Figure: $P\left(\overline{A}\right)$ from Wikipedia

Example

• The birthday problem occurs in disguise in many places. For example in the german lottery (similar to Powerball) where one picks 6 numbers out 49 numbers the exact same numbers 15, 25,27,30,42,48 appeared **twice** in December 1988 and June 1995?
It looks like an amazing coincidence since the probability to pick those number is $\frac{1}{\binom{49}{6}} = \frac{1}{13{,}986{,}816}$
However the question is: this lottery had run by then for 29 years and 3016 biweekly draws. How likely is it that the same numbers appear twice is a "birthday" problem.

The binomial theorem

Theorem

The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof:

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

Expanding the product gives a sum of 2^n terms. Each term has the form $x^k y^{n-k}$ with $k = 0, 1, \cdots, n$.
The number of terms of the form $x^k y^{n-k}$ is equal to $\binom{n}{k}$ since it is the number of ways to choose the k locations for the x.
Adding all the terms up give the theorem.

Multinomial coefficients

If you split 10 people in 3 groups of size 5, 3 and 2 there are

$$\underbrace{\binom{10}{5}}_{\text{pick } 5}\underbrace{\binom{5}{3}}_{\text{pick } 3}\underbrace{\binom{2}{2}}_{\text{pick } 2} = \frac{10!}{5!\cancel{5!}}\frac{\cancel{5!}}{3!\cancel{2!}}\frac{\cancel{2!}}{2!\cancel{0!}} = \frac{10!}{5!3!2!}$$

ways to do it. More generally

Multinomial coefficients

There are

$$\binom{n}{n_1\, n_2 \cdots n_r} = \frac{n!}{n_1!n_2!\cdots n_r!} \quad \text{with } n_1 + n_2 + \cdots n_r = n$$

possible ways to divide n items into r groups of respective size n_1, n_2, $\cdots$ n$_r$.