STAT 315: Order Statistics

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December 3, 2025

General Idea: Using CDFs

Let $X_1, \ldots, X_n$ be real-valued random variables and define

$$M_n = \max\{X_1, \ldots, X_n\}, \qquad m_n = \min\{X_1, \ldots, X_n\}.$$

- The key tool is the distribution function (CDF):

$$F_{X_i}(x) = \mathbb{P}(X_i \leq x).$$

- To find the distribution of M_n or m_n, we express events like $\{M_n \leq x\}$ and $\{m_n \leq x\}$ in terms of the X_i.
- Often we assume the X_i are independent and identically distributed (i.i.d.), which simplifies the formulas.

Distribution of the Maximum of Two Variables

Let X and Y be random variables and define

$$M = \max\{X, Y\}.$$

- By definition:

$$F_M(x) = \mathbb{P}(M \leq x) = \mathbb{P}(\max\{X, Y\} \leq x).$$

- The event $\{\max\{X, Y\} \leq x\}$ is the same as

$$\{X \leq x \text{ and } Y \leq x\}.$$

- Hence

$$F_M(x) = \mathbb{P}(X \leq x,\ Y \leq x).$$

- If X and Y are independent, then

$$F_M(x) = \mathbb{P}(X \leq x)\,\mathbb{P}(Y \leq x) = F_X(x)\,F_Y(x).$$

Distribution of the Minimum of Two Variables

Let X and Y be random variables and define

$$m = \min\{X, Y\}.$$

- We start with

$$F_m(x) = \mathbb{P}(m \leq x) = \mathbb{P}(\min\{X, Y\} \leq x).$$

- The complement event is

$$\{\min\{X, Y\} > x\} = \{X > x \text{ and } Y > x\}.$$

- Therefore

$$F_m(x) = 1 - \mathbb{P}(X > x,\ Y > x).$$

- If X and Y are independent,

$$\mathbb{P}(X > x,\ Y > x) = \mathbb{P}(X > x)\,\mathbb{P}(Y > x) = (1 - F_X(x))(1 - F_Y(x)).$$

$$\Rightarrow \quad F_m(x) = 1 - (1 - F_X(x))(1 - F_Y(x)).$$

i.i.d. Case: Maximum and Minimum of n Variables

Assume $X_1, \ldots, X_n$ are i.i.d. with CDF $F(x) = \mathbb{P}(X_i \leq x)$.

Maximum: $M_n = \max\{X_1, \ldots, X_n\}$

$$F_{M_n}(x) = \mathbb{P}(M_n \leq x) = \mathbb{P}(X_1 \leq x, \ldots, X_n \leq x) = [F(x)]^n.$$

If X_i are continuous with density f, then the density of M_n is

$$f_{M_n}(x) = \frac{d}{dx} F_{M_n}(x) = n[F(x)]^{n-1} f(x).$$

Minimum: $m_n = \min\{X_1, \ldots, X_n\}$

$$\mathbb{P}(m_n > x) = \mathbb{P}(X_1 > x, \ldots, X_n > x) = [1 - F(x)]^n.$$

$$\Rightarrow \quad F_{m_n}(x) = \mathbb{P}(m_n \leq x) = 1 - [1 - F(x)]^n.$$

If X_i are continuous,

$$f_{m_n}(x) = \frac{d}{dx} F_{m_n}(x) = n[1 - F(x)]^{n-1} f(x).$$

Example: Max and Min of Uniform(0,1)

Let $X_1, \ldots, X_n$ be i.i.d. Uniform$(0,1)$, so $F(x) = x$ on $(0,1)$.

Maximum: $M_n = \max\{X_1, \ldots, X_n\}$

$$F_{M_n}(x) = x^n, \quad 0 < x < 1,$$

$$f_{M_n}(x) = nx^{n-1}, \quad 0 < x < 1.$$

Minimum: $m_n = \min\{X_1, \ldots, X_n\}$

$$F_{m_n}(x) = 1 - (1-x)^n, \quad 0 < x < 1,$$

$$f_{m_n}(x) = n(1-x)^{n-1}, \quad 0 < x < 1.$$

- These are Beta distributions:

$$M_n \sim \mathrm{Beta}(n, 1), \qquad m_n \sim \mathrm{Beta}(1, n).$$

Example: Min and Max of Exponential Variables

Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{Exp}(\lambda)$ random variables with

$$F(x) = 1 - e^{-\lambda x}, \qquad f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

Minimum: $m_n = \min\{X_1, \ldots, X_n\}$

$$\mathbb{P}(m_n > x) = \mathbb{P}(X_1 > x, \ldots, X_n > x) = (e^{-\lambda x})^n = e^{-n\lambda x}.$$

Thus

$$F_{m_n}(x) = 1 - e^{-n\lambda x}, \qquad f_{m_n}(x) = n\lambda e^{-n\lambda x}.$$

Conclusion: The minimum is exponential: $m_n \sim \mathrm{Exp}(n\lambda)$ with n times the rate.

Maximum: $M_n = \max\{X_1, \ldots, X_n\}$

$$F_{M_n}(x) = [F(x)]^n = (1 - e^{-\lambda x})^n, \qquad x \geq 0.$$

Density: $f_{M_n}(x) = n(1 - e^{-\lambda x})^{n-1} \lambda e^{-\lambda x}$ (called Beta–Weibull).

Example: Min and Max of Two Fair Dice

Let X and Y be the outcomes of two independent fair dice, each uniform on $\{1, 2, \ldots, 6\}$. Define $M = \max(X, Y), m = \min(X, Y)$.

Distribution of the maximum.

$$F_M(k) = \mathbb{P}(M \le k) = \mathbb{P}(X \le k,\ Y \le k) = \left(\frac{k}{6}\right)^2, \qquad k = 1, \ldots, 6.$$

$$\mathbb{P}(M = k) = F_M(k) - F_M(k-1) = \frac{k^2 - (k-1)^2}{36} = \frac{2k-1}{36}.$$

Distribution of the minimum.

$$\mathbb{P}(m > k) = \mathbb{P}(X > k,\ Y > k) = \left(\frac{6-k}{6}\right)^2.$$

$$F_m(k) = \mathbb{P}(m \le k) = 1 - \left(\frac{6-k}{6}\right)^2, \qquad k = 1, \ldots, 6,$$

$$\mathbb{P}(m = k) = F_m(k) - F_m(k-1) = \frac{(6-k+1)^2 - (6-k)^2}{36} = \frac{13-2k}{36}.$$