

STAT 315: Moment generating functions

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Moment generating functions of a random variable X

So far we have seen two ways to describe a RV

- The PDF $p(x)$ (X discrete) or $f(x)$ (X continuous).
- The CDF $F(X) = P(X \leq x)$.

These are useful to compute

$$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$$

and so on.

We introduce another way, more indirect way, to describe a RV X , the **moment generating function of X $m(t)$** .. This is a function of a real variable t given by

$$t \mapsto m(t) = E[e^{tX}]$$

The MGF = moment generating function

Moment generating functions

For a random variable the **moment generating function** $m(t)$ (**abbreviated MGF**) is given by

$$m(t) = E[e^{tY}] = \sum_y e^{ty} p(y) \quad \text{discrete random variable}$$

$$m(t) = E[e^{tY}] = \int_{-\infty}^{\infty} e^{ty} f(y) dy \quad \text{continuous random variable}$$

Example: Consider a Bernoulli RV Y (that is a binomial random variables with parameters $n = 1$ and p and PDF $p(0) = 1 - p$ and $p(1) = p$. Then the MGF is

$$m(t) = E[e^{tY}] = \sum_x e^{tx} p(x) = p(0)e^0 + p(1)e^t = (1 - p) + pe^t$$

Uniqueness of MGF

The usefulness of MGF comes from the following theorem (which is not easy to prove at all!)

Theorem (Uniqueness)

*The moment generating function $m(t)$ **determines a random variable uniquely.***

More precisely suppose X and Y have moment generating function $m_X(t)$ and $m_Y(t)$ which are finite for t around 0. Then

$$m_X(t) = m_Y(t) \text{ implies } X = Y.$$

We will use that in Section 6!

MGF for binomial RV

- The Binomial Random variable has pdf $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ and so the MGF is

$$\begin{aligned} m(t) &= E[e^{tX}] = \sum_k e^{tk} p(k) = \sum_k e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_k \binom{n}{k} (e^t p)^k (1-p)^{n-k} = (pe^t + (1-p))^n \end{aligned}$$

using the binomial theorem $(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$

- The previous theorem tells the converse: if $m_X(t) = (.2e^t + .8)^5$ then X is binomial with parameter $n = 5$ and $p = .2$ and so for example $P(X = 3) = \binom{5}{3} (0.2)^3 (0.8)^2$

Moments from MGF

Moments

If Y is a random variable with MGF $m(t)$ then its moments are

$$E[Y^k] = m^{(k)}(0) \quad k^{\text{th}} \text{ derivative evaluated at } 0$$

Proof: For continuous random variables

$$m'(t) = \frac{d}{dt} \int e^{ty} f(y) dy = \int \frac{d}{dt} e^{ty} f(y) dx = \int y e^{ty} f(y) dy = E[Y e^{tY}]$$

and so $m'(0) = \int y f(y) dy = E[Y]$. Similarly

$$m^{(k)}(t) = \int \frac{d^k}{dt^k} e^{ty} f(y) dy = \int y^k e^{ty} f(y) dy = E[Y^k e^{tY}]$$

and so $m^{(k)}(0) = E[Y^k]$.

MGF for the Gamma random variables

MGF for Gamma

The MGF for a gamma RV is given by

$$m(t) = E[e^{tY}] = \frac{1}{(1 - \beta t)^\alpha} = (1 - \beta t)^{-\alpha}$$

Proof:

$$\begin{aligned} m(t) &= E[e^{tY}] = \int_0^\infty e^{ty} \frac{y^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} dy = \int_0^\infty \frac{y^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\left(\frac{1}{\beta} - t\right)y} dy \\ &= \frac{\beta(t)^\alpha}{\beta^\alpha} \underbrace{\int_0^\infty \frac{y^{\alpha-1}}{\beta(t)^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta(t)}} dy}_{=1} \quad \text{with } \frac{1}{\beta(t)} = \frac{1}{\beta} - t = \frac{(1 - \beta t)}{\beta} \\ &= \frac{1}{(1 - \beta t)^\alpha} \end{aligned}$$

Moments for the Gamma random variable

Since

$$m(t) = (1 - \beta t)^{-\alpha}$$

we find

$$m'(t) = \alpha\beta(1 - \beta t)^{-\alpha-1}$$

$$m''(t) = \alpha(\alpha + 1)\beta^2(1 - \beta t)^{-\alpha-2}$$

$$m'''(t) = \alpha(\alpha + 1)(\alpha + 2)(\beta - t)^{-\alpha-3}$$

We find

$$E[Y] = \alpha\beta, \quad E[Y^2] = \alpha(\alpha + 1)\beta^2$$

and so the variance is

$$V(Y) = \alpha(\alpha + 1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

Linear transformations

If Y is a random variable and a and b are constants.

Mean: $E[aY + b] = aE[Y] + b$

Variance: $V[aY + b] = a^2 V[Y]$

MGF: $m_{aY+b}(t) = E[e^{t(aY+b)}] = e^{bt} E[e^{atY}] = e^{bt} m_Y(at)$

Example: Suppose Y is exponential with parameter 2 so the MGF is $m(t) = (1 - 2t)^{-1}$.

Let $Z = 3Y$ then and so

$$m_Z(t) = E[e^{tZ}] = E[e^{3tY}] = m_Y(3t) = (1 - 2 \times 3t)^{-1} = (1 - 6t)^{-1}$$

By the uniqueness theorem Z is exponential with parameter 6.

MGF for normal RV, part 1

- **Normalization of the standard normal**

Use polar coordinate

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) \\&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy \\&= \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2} r dr d\theta \quad \text{polar coordinates} \\&= 2\pi \int_0^{\infty} e^{-s} ds \quad \text{change of variable } s = r^2/2 \\&= 2\pi\end{aligned}$$

and thus

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

MGF for normal RV, part 2

- MGF for the standard normal Z Use that (complete the square)

$$\frac{x^2}{2} - xt = \left(\frac{x^2}{2} - xt + \frac{t^2}{2} \right) - \frac{t^2}{2} = \frac{(x - t)^2}{2} + \frac{t^2}{2}$$

and so

$$\begin{aligned} m_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t^2/2} e^{-(x-t)^2/2} dx \\ &= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \quad y = x - t \\ &= e^{t^2/2} \end{aligned}$$

Thus

$$m_Z(t) = e^{t^2/2}$$

MGF for normal RV, 3

- MGF for the normal Y

With the change of variable $y = (x - \mu)/\sigma$ and $dy = dx/\sigma$

$$\begin{aligned}m_Y(t) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{xt} e^{-(x-\mu)^2/2\sigma^2} dx \\&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma y)} e^{-y^2/2} dy \\&= e^{t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{t\sigma y} e^{-y^2/2} dy \\&= e^{t\mu} m_Z(t\sigma) \\&= e^{t\mu + \sigma^2 t^2/2}\end{aligned}$$

- By the uniqueness theorem this also proves that

$$Y \sim N(\mu, \sigma^2) \implies Z = \frac{Y - \mu}{\sigma} \sim N(0, 1)$$

$$Z \sim N(0, 1) \implies Y = \sigma Z + \mu \sim N(\mu, \sigma^2)$$

Example of MGF

	PDF	CDF	MGF
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	...	$(pe^t + (1-p))^n$
Geometric	$(1-p)^{k-1} p$	$1 - (1-p)^k$	$\frac{pe^t}{1-(1-p)e^t}$
Poisson	$e^{-\lambda} \frac{\lambda^k}{k!}$		$e^{\lambda(e^t-1)}$
Exponential	$\lambda e^{-\lambda t}$	$1 - e^{-\lambda t}$	$\frac{\lambda}{\lambda-t}$
Gamma	$\frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}$	...	$\frac{1}{(1-\beta t)^\alpha}$
Normal	$\frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$		$e^{\mu t + \sigma^2 t^2/2}$
Uniform	$\frac{1}{\theta_2 - \theta_1}$	$\frac{y - \theta_1}{\theta_1 - \theta_1}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_1 - \theta_1)}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1} (1-y)^{\beta-1}$		

CDF calculator: <https://homepage.divms.uiowa.edu/~mbognar/> or google colab