STAT 315: Mean, variance, and covariance for discrete joint RV

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Expected value of function of joint random variables

If Y_1, Y_2 are joint RV and $g : \mathbb{R}^2 \to \mathbb{R}$ is a function then we can compute the expected value of $g(Y_1, Y_2)$

Expected value(discrete

For joint random discrete variables Y_1 and Y_2 with joint pdf $p(y_1, y_2)$ and a function $g(Y_1, Y_2)$ we have

$$E[g(Y_1, Y_2)] = \sum_{y_1, y_2} g(y_1, y_2) p(y_1, y_2) \quad \text{discrete RV}$$

Expected value (continuous

For joint random continuous variables Y_1 and Y_2 with joint pdf $f(y_1, y_2)$ and a function $g(Y_1, Y_2)$ we have

$$E[g(Y_1, Y_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1, y_2) f(y_1, y_2)\, dy_1 \, dy_2 \quad \text{continuous RV}$$

Linearity of expected value

Linearity

- For any constant c
$$E[c] = c$$

- For any function $g(Y_1, Y_2)$ and any constant c
$$E[c\, g(Y_1, Y_2)] = c\, E[g(Y_1, Y_2)]$$

- For any functions $g(Y_1, Y_2)$ and $h(Y_1, Y_2)$
$$E[g(Y_1, Y_2) + h(Y_1, Y_2)] = E[g(Y_1, Y_2)] + E[h(Y_1, Y_2)]$$

Same proof as for $f(Y)$!

Independence and products

Independence and products

If Y_1 and Y_2 are independent then for any functions $g(Y_1)$ and $h(Y_2)$

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)]$$

For example independence implies that have

$$E[Y_1 Y_2] = E[Y_1]E[Y_2]$$

Proof: Independence means $p(y_1, y_2) = p(y_1)p(y_2)$ and so

$$\begin{aligned} E[g(Y_1)h(Y_2)] &= \sum_{y_1,y_2} g(y_1)h(y_2)p(y_1)p(y_2) \\ &= \sum_{y_1} g(y_1)p(y_1) \sum_{y_2} h(y_2)p(y_2) \\ &= E[g(Y_1)]E[h(Y_2)] \end{aligned}$$

Covariance

Covariance of Y_1 and Y_2

If Y_1 and Y_2 are random variables with means $\mu_1 = E[Y_1]$ and $\mu_2 = E[Y_2]$ then the covariance of Y_1 and Y_2 is

$$\mathrm{Cov}(Y_1, Y_2) = E\left[(Y_1 - \mu_1)(Y_2 - \mu_2)\right]$$

and the correlation coefficient ρ is

$$\rho = \rho(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_1 \sigma_2}$$

We say that Y_1 and Y_2 are

- positively correlated if $Cov(Y_1, Y_2) > 0$
- negatively correlated if $Cov(Y_1, Y_2) < 0$
- uncorrelated if $Cov(Y_1, Y_2) = 0$

Properties of covariance

1. We have the formula

$$\mathrm{Cov}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$$

2. $\mathrm{Cov}(Y_1, Y_1) = V(Y_1)$ and so $\rho(Y_1, Y_1) = 1$

3. We have Cauchy-Schwartz inequality

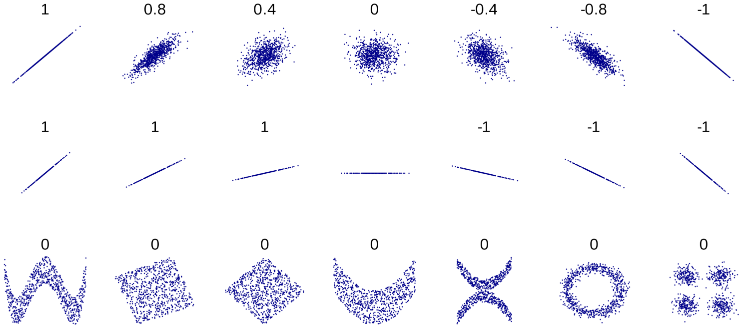
$$|E[Z_1 Z_2]| \leq \sqrt{E[Z_1^2]E[Z_2^2]}$$

   and as a consequence the correlation coefficient satisfies

$$-1 \leq \rho \leq 1$$

4. If Y_1 and Y_2 are independent then $\mathrm{Cov}(Y_1, Y_2) = 0$ and so Y_1 and Y_2 are uncorrelated.
   But the converse is not always true

Example of correlation coefficients



Correlation capture the linear dependence between RV (but not non-linear dependences) (third row)

The correlation reflects the noisiness and direction of a linear relationship (top row), but not the slope of that relationship (second row)

Image taken from Wikipedia

Example: Discrete Random Variables

Story: A professor records how long students study (X) and whether they pass an exam (Y).

- $X \in \{0, 1, 2\}$: number of hours studied
- $Y \in \{0, 1\}$: pass (1) or fail (0)

Joint Probability Distribution:

X	$Y = 0$	$Y = 1$	$p_X(x)$
0	0.30	0.05	0.35
1	0.10	0.20	0.30
2	0.05	0.30	0.35
$p_Y(y)$	0.45	0.55	1

Step 1: Expectations

$$\mathbb{E}[X] = 0(0.35)+1(0.30)+2(0.35) = 1.0, \quad \mathbb{E}[Y] = 0(0.45)+1(0.55) = 0.55.$$

Step 2: Mixed Moment

$$\mathbb{E}[XY] = 0(0.05) + 0(0.30) + 1(0.20) + 2(0.30) = 0.80.$$

Step 3: Covariance

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.80 - (1.0)(0.55) = \boxed{0.25.}$$

Interpretation: A positive covariance (0.25) indicates that students who study more tend to have higher probability of passing. If study time and success were independent, covariance would be 0.

Examples

- For the joint RV (X, Y) with PDF $f(x, y) = 2e^{-2x}e^{-y}, x \geq 0, y \geq 0$ compute $\mathrm{Cov}(\mathrm{X}, \mathrm{Y})$.
- For gas tank problem with the joint RV (Y_1, Y_2) with joint PDF $f(y_1, y_2) = 3y_1$ for
$$0 \leq y_2 \leq y_1 \leq 1$$
compute $\mathrm{Cov}(\mathrm{X}, \mathrm{Y})$ and the correlation coefficient $\rho(Y_1, Y_2)$. Compute the mean and the variance of $Y_1 - Y_2$ (the quantity of unsold gas).
- Suppose Y_1 and Y_2 be the proportion of two chemical in a mixture so that we must have $Y_1 + Y_2 \leq 1$. We take the PDF to be
$$f(y_1, y_2) = 2 \quad \text{if } , y_1 \geq 0, y_2 \geq 0\,, y_1 + y_2 \leq 1$$
Compute $\mathrm{Cov}(Y_1, Y_2)$ and the correlation coefficient $\rho(Y_1, Y_2)$

Correlation versus independence

- Take X uniform on $[-1, 1]$ and $Y = X^2$. Are X and Y independent? Compute $\mathrm{Cov}(X, Y)$.

- Consider the discrete RVs with joint PDF

$X \backslash Y$	-1	0	1	$P_X(x)$
-1	$1/16$	$3/16$	$1/16$	$5/16$
0	$3/16$	0	$3/16$	$6/16$
1	$1/16$	$3/16$	$1/16$	$5/16$
$P_Y(y)$	$5/16$	$6/16$	$5/16$	

Are X and Y independent?

Compute $\mathrm{Cov}(X, Y)$.

Linear combinations of random variables

Linear combination

For random variables Y_1, Y_2 and Z_1, Z_2 and constants a_1, a_2 and b_1, b_2.

Expected Value

$$E[a_1 Y_1 + a_2 Y_2] = a_1 E[Y_1] + a_2 E[Y_2]$$

Variance

$$V(a_1 Y_1 + a_2 Y_2) = a_1^2 V(Y_1) + a_2^2 V(Y_2) + 2 a_1 a_2 \mathrm{Cov}(Y_1, Y_2)$$

Covariance

$$\mathrm{Cov}(a_1 Y_1 + a_2 Y_2, b_1 Z_1 + b_2 Z_2) = a_1 b_1 \mathrm{Cov}(Y_1, Z_1) + \\ + a_1 b_2 \mathrm{Cov}(Y_1, Z_2) + a_2 b_1 \mathrm{Cov}(Y_2, Z_1) + a_2 b_2 \mathrm{Cov}(Y_2, Z_2)$$

Practice: Variance, Covariance, and Correlation

1. X, Y are uncorrelated with $\mathrm{Var}(X) = 4$, $\mathrm{Var}(Y) = 9$. Compute $\mathrm{Var}(2X - 3Y)$.
2. X, Y have $\mathrm{Var}(X) = 1$, $\mathrm{Var}(Y) = 4$, $\mathrm{Cov}(X, Y) = 1$. Compute:
   (a) $\mathrm{Var}(X + Y)$
   (b) $\mathrm{Var}(2X - Y)$
3. Given $\mathrm{Var}(X + Y) = 25$, $\mathrm{Var}(X) = 9$, $\mathrm{Var}(Y) = 16$, find $\mathrm{Cov}(X, Y)$.
4. X_1, X_2, X_3 are uncorrelated with $\mathrm{Var}(X_1) = 1$, $\mathrm{Var}(X_2) = 2$, $\mathrm{Var}(X_3) = 3$. Compute $\mathrm{Var}(2X_1 - X_2 + 3X_3)$.
5. X, Y, Z have $\mathrm{Var}(X) = \mathrm{Var}(Y) = \mathrm{Var}(Z) = 1$ and $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, Z) = \mathrm{Cov}(Y, Z) = \rho$. Find $\mathrm{Var}(X + Y + Z)$ in terms of ρ.
6. X, Y have $\mathrm{Var}(X) = 9$, $\mathrm{Var}(Y) = 4$, $\mathrm{Cov}(X, Y) = -3$. Compute $\mathrm{Var}(X + 2Y)$ and discuss the effect of negative covariance.
7. (With correlation) $E[X] = E[Y] = 0$, $\mathrm{Var}(X) = 4$, $\mathrm{Var}(Y) = 9$, $\rho(X, Y) = 0.5$. Compute:
   (a) $\mathrm{Cov}(X, Y)$
   (b) $\mathrm{Var}(2X - Y)$
   (c) $\rho(2X - Y, X)$

Diversify to minimize risk

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You are given 2 investments X_1 and X_2 which are independent, say
$X_1 =$ crypto and $X_2 =$ mongolian sheeps.

Assume that

$$E[X_1] = E[X_2] \quad \text{same average return}$$

$$V(X_1) = \sigma_1^2 \qquad V(X_2) = \sigma_2^2 \quad \text{different risks}$$

: Goal is to minimize the risk of your investment: What should you do?

Diversify: Allocate proportion α to X_1 and $(1-\alpha)$ to X_2 with $0 \leq \alpha \leq 1$

$$X = \alpha X_1 + (1-\alpha) X_2$$

$$V(X) = \alpha^2 V(X_1) + (1-\alpha)^2 V(X_2) = \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2$$

Minimize with respect to α:

$$0 = 2\alpha\sigma_1^2 - 2(1-\alpha)\sigma_2^2 \implies \alpha = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

If $\sigma_1^2 = 10$ (crypto) and $\sigma_2^2 = 1$(sheep) you should invest $\alpha^* = \frac{1}{11}$ in crypto and $(1-\alpha^*) = \frac{10}{11}$ in sheep.

Then the optimal variance is

$$V(X) = \left(\frac{1}{11}\right)^2 \times 10 + \left(\frac{10}{11}\right)^2 \times 1 = \frac{110}{121} < 1$$

Don't put all your eggs in the same basket.

Mean and Variance of sample averages

Empirical or sample average

Suppose $Y_1, Y_2, \cdots Y_n$ are independent random variables with

$$E[Y_i] = \mu \qquad V(Y_1) = \sigma^2$$

Then

$$E\left[\frac{Y_1 + Y_2 + \cdots Y_n}{n}\right] = \mu$$

and

$$V\left(\frac{Y_1 + Y_2 + \cdots Y_n}{n}\right) = \frac{\sigma^2}{n}$$

Very important for later