STAT 315: Joint PDF of continuous random variables

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October 29, 2025

Bivariate or joint random variables

Suppose we have random experiment and make TWO measurements Y_1 and Y_2 or more....

Examples:

- We measure the height and weight of some individual in a population.
- We have an exponential random variable whose parameter β is itself random and obeys a certain distribution.
- We throw a dart at a random position (Y_1, Y_2) on a circular target.
- $\cdots$
- Sampling: We repeat an experiment n times and record the results $Y_1, Y_2, \cdots, Y_n$ of the experiments (the most important example!)

We need to describe the probability distribution of Y_1 and Y_2 together! This is called the joint (or bivariate) PDF $f(y_1, y_2)$ (continuous).

Joint PDF of continuous

Joint (or bivariate) PDF for continuous random variables

The joint continuous RV (Y_1, Y_2) have joint PDF $f(y_1, y_2)$ if

$$P(a_1 \leq Y_1 \leq b_1, a_2 \leq Y_2 \leq b_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(y_1, y_2) dy_1, dy_2$$

$$\text{with } 0 \leq f(y_1, y_2) \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1$$

Example 1: The random variables (X, Y) have the joint PDF

$$f(x, y) = \begin{cases} 2e^{-2x} e^{-y} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{else} \end{cases}$$

Example 2: A gas station adds some gas to its tank every Monday morning. We write Y_1 (in $[0,1]$) for the proportion of the tank being filled and Y_2 for the proportion of the tank which is sold to customers during the subsequent. Note that we must have $Y_2 \leq Y_1$. We propose the model

$$f(y_1, y_2) = \begin{cases} 3y_1 & \text{if } 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{else} \end{cases}$$

Example 3: Two friends, independently of each other, arrive at a random time between 12pm and 1pm at the blue wall. We can describe the time of their arrival by two random variable Y_1, Y_2 each with a uniform RV on $[0,1]$ (measured in hours). The independence assumption leads to the model

$$f(y_1, y_2) = \begin{cases} 1 & \text{if } 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{else} \end{cases}$$

Example 4: A (pretty bad) player throws a dart at a random point on a circular target of radius R. This can be described by a uniform distribution on a disk of radius R that is by the joint random variables (X_1, X_2) with pdf

$$f(x_1, x_2) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x_1^2 + x_2^2 \leq R^2 \\ 0 & \text{else} \end{cases}$$

Example 5: In Bayesian statistics context one uses random variables whose parameters are themselves random variables. For example consider the joint PDF

$$f(x, y) = \begin{cases} ye^{-yx}e^{-y} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{else} \end{cases}$$

As we will see this describe a an exponential random variable whose scale parameters (i.e. with pdf $\lambda e^{-\lambda x}$) has a exponential distribution with parameter 1.

Marginal and conditional PDF

Marginal PDF of continuous random variables

If the joint continuous RV (Y_1, Y_2) has PDF $f(y_1, y_2)$ then the marginal PDFs of Y_1 and Y_2 are given by

$$f(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \qquad f(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1$$

Conditional PDF of continuous random variables

If the joint continuous RV (Y_1, Y_2) has PDF $p(y_1, y_2)$ then the conditional PDFs of Y_1 given $Y_2 = y_2$ is given by

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f(y_2)}$$

if $f(y_2) > 0$

Independence

Recall that the events A and B are independent if

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B) \text{ or } P(A \cap B) = P(A)P(B)$$

Independence of continuous random variables

The continuous random variables Y_1 and Y_2 are independent if

$$f(y_1|y_2) = f(y_1) \text{ or } f(y_2|y_1) = f(y_2) \text{ or } f(y_1, y_2) = f(y_1)f(y_2)$$

Criterion for independence

The random variables Y_1 and Y_2 are independent if and only if

$$f(y_1, y_2) = g(y_1)h(y_2) \quad -\infty < y_1, y_2 < \infty$$

for some function $g(x)$ and $h(y)$

Examples

The questions refer to Example 1–5 introduced at the beginning of the slides

- For each proposed PDF make sure they are normalized
- Compute
  1. Example 1: $P(X \geq 1, Y \leq 2)$ and $P(X \leq Y)$.
  2. Example 2: $P(Y_1 \leq 3/4, Y_2 \geq 1/2)$
  3. Example 3: What the probability that the two friends wait more than 15 minutes for each other?
  4. Example 4: If $R = 1$ what is the probability that the dart does not land within .1 of the center of the target.
  5. Example 5: $P(X \geq 1, Y \geq 2)$.
- Which ones are independent?
- Find the marginals and conditionals PDF in each example.