STAT 315: Joint discrete random variables

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Bivariate or joint random variables

Suppose we perform a random experiment. A random variable Y is obtained by performing ONE measurement on the experiment. Joint (or bivariate) random variables Y_1 and Y_2 are obtained by performing TWO simultaneous measurements on the experiment.

Examples:

- We measure the height and weight of some individual in a population.
- We roll three dice and are interested in the number of 1 rolled and the total sum of the dice.
- ......
- Sampling: We repeat an experiment n times and record the results $Y_1, Y_2, \cdots, Y_n$ of the experiments (the most important example!)

We need to describe the probability distribution of Y_1 and Y_2 together! This is called the joint (or bivariate) PDF $p(y_1, y_2)$ (discrete)

Joint PDF of discrete RVs.

Joint (or bivariate) PDF for random variables

The joint discrete RV (Y_1, Y_2) have joint a PDF $p(y_1, y_2)$ if

$$P(Y_1 = y_1, Y_2 = y_2) = p(y_1, y_2)$$

$$\text{with } 0 \le p(y_1, y_2) \le 1 \quad \text{and} \quad \sum_{y_1, y_2} p(y_1, y_2) = 1$$

- Bivariate (=statistics) vs joint (=probability)
- Generalize to more than 2 RV, multivariate RVs.

Joint PDF as table

- 15% of families have 0 children, 20% of families have 1 children, 35% of families have 2 children, 30% of families have 3 children
- boys and girls are equally likely.
- Pick a family at random and let $B =$ number of boys and $G =$ number of girls

		B			
		0	1	2	3
G	0	.15	.1	.0875	.0375
	1	.1	.175	.1125	
	2	.0875	.1125		
	3	.0375			

Compute all the entries!

Marginal PDF

We can recover the PDF of Y_1 and Y_2 from the joint PDF. This corresponding to summming rows and columns

Marginal PDF

If the joint discrete RV (Y_1, Y_2) has PDF $p(y_1, y_2)$ then the marginal PDFs of Y_1 and Y_2 are given by

$$p(y_1) = \sum_{y_2} p(y_1, y_2) \qquad p(y_2) = \sum_{y_1} p(y_1, y_2)$$

		B				
		0	1	2	3	
G	0	.15	.1	.0875	.0375	.375
	1	.1	.175	.1125		.3875
	2	.0875	.1125			.2
	3	.0375				.0375
		.375	.3875	.2	.0375	

Conditional PDF

Conditional PDF

The conditional PDFs of Y_1 given $Y_2 = y_2$ is given by

$$P(Y_1 = y_1 | Y_2 = y_2) = p(y_1|y_2) = \frac{p(y_1, y_2)}{p(y_2)}$$

We write $Y_1|Y_2 = y_2$ for the RV Y_1 conditioned on $Y_2 = y_2$

		B				
		0	1	2	3	
G	0	.15	.1	.0875	.0375	.375
	1	.1	.175	.1125		.3875
	2	.0875	.1125			.2
	3	.0375				.0375
		.375	.3875	.2	.0375	

$G\|B = 2$
.4375
.5625
.0
.0

Independence

- Recall that the events A and B are independent if

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B) \text{ or } P(A \cap B) = P(A)P(B)$$

- Two RV Y_1 and Y_2 are independent if knowing the value for 1 RV one does not influence the probabilities of the other one.

Independence of random variables

The discrete random variables Y_1 and Y_2 are independent if

$$\begin{aligned} & p(y_1|y_2) = p(y_1) \\ \text{or } & p(y_2|y_1) = p(y_2) \\ \text{or } & p(y_1, y_2) = p(y_1)p(y_2) \end{aligned}$$

Example

B and G are not independent (clearly!)

		B			
		0	1	2	3
G	0	.15	.1	.0875	.0375
	1	.1	.175	.1125	
	2	.0875	.1125		
	3	.0375			

$$p(2,2) = 0 \neq p(2)p(2)$$

More example

- Two customers enters a store which has 3 cash registers. They both pick a registers at random to check out.
  - Find the joint distribution of register 1 and 2 denoted by (Y_1, Y_2).
  - Find the marginal distribution of Y_1 and Y_2?
  - Are Y_1 and Y_2 independent?
  - Find the conditional distribution of Y_2 given $Y_1 = 0, 1, 2$?
- Toss three coins successively and consider the random variables
  Y_1=number of Heads obtained
  $Y = 2$ is a side be and takes the value 1 if the first head is on the first coin, value 2 if the first head on the second coin and 3 if the first head on the second coin. If there is no head you lose 1.
  - Find the joint distribution of Y_1 and Y_2.
  - Marginal distributions of Y_1 and Y_2?