STAT 315: Exponential and Gamma Random Variables

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Exponential Random Variables

Exponential Random Variables

A random variable Y is an exponential random variable with parameters β and we write $Y \sim Exp(\beta)$ if the PDF is

$$f(y) = \frac{1}{\beta} e^{-\frac{y}{\beta}} \qquad \beta > 0 \text{ scale parameter}$$

The CDF is given by

$$F(y) = \int_0^y \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = 1 - e^{-\frac{y}{\beta}}$$

Warning: Very often $\lambda = 1/\beta$ is used a parameter with

$$f(y) = \lambda e^{-\lambda y} \qquad \lambda > 0 \text{ rate parameter}$$

PDF and CDF

PDF: $f(t) = \frac{1}{\beta} e^{-\frac{y}{\beta}}$ CDF: $F(t) = 1 - e^{-\frac{y}{\beta}}$

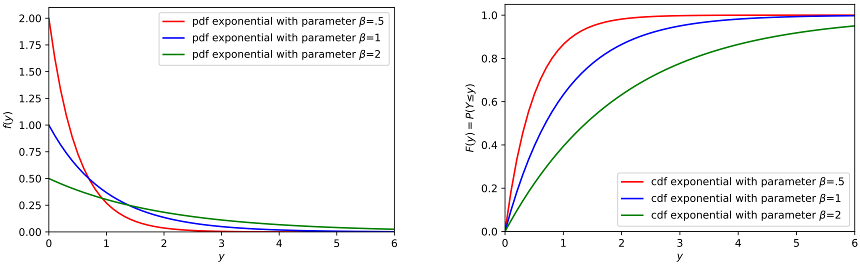


Figure: The Exponential RV with Left:PDF and Right:CDF

Waiting times and the memoryless property

Often Y has the interpretation of a waiting time

- Y is the time between two consecutive earthquake in California
- Y is the time between the emission of two radioactive particles.
- The time it takes to be served at a cash register at a supermarket.
- $\cdots$

Memoryless property

For an exponential random variable Y

$$P(Y \geq t + s | Y \geq t) = P(Y \geq s)$$

"If you have waited for at least 1 hours, the probability you have to wait another 30 minutes is the same as if you just had arrived...."

Mean and Variance of Exponential RV

Moments of exponential RV

For a exponential random variable Y with parameter and β we have

$$E[Y] = \beta \qquad V(Y) = \beta^2$$

Proof: Integration by parts. See the more general computation later for Gamma random variables

Gamma Random Variable

Gamma random variables

A gamma random variables Y with parameters α (shape parameter) and β (scale parameter) (we write $Y \sim \Gamma(\alpha, \beta)$) has the pdf

$$f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)} \quad \alpha > 0, \beta > 0$$

Gamma function

The gamma function $\Gamma(\alpha)$ is given by

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

It satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(n) = (n-1)!$ and $\Gamma(1/2) = \sqrt{\pi}$

Properties of Gamma function

- Integration by parts with $u = y^\alpha$ and $v' = e^{-y}$

$$\Gamma(\alpha+1) = \int_0^\infty y^\alpha e^{-y}\,dy = -y^\alpha e^{-y}|_{-\infty}^{\infty} + \alpha\int_0^\infty y^{\alpha-1}e^{-y}\,dy = \alpha\Gamma(\alpha)$$

- We have $\Gamma(1) = \int_0^\infty e^{-y}\,dy = 1$ and so

$$\Gamma(n) = (n-1)\Gamma(n-2) = (n-1)(n-2)\Gamma(n-3) = \cdots = (n-1)!$$

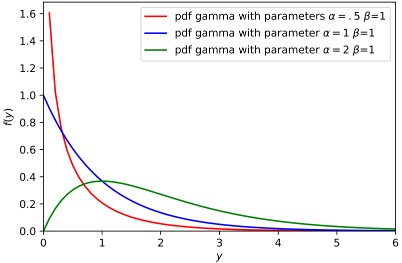
- Using the change of variable $y = \frac{x^2}{2}$, $dy = xdx$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty y^{-\frac{1}{2}}e^{-y}\,dy = \int_0^\infty \frac{\sqrt{2}}{x}e^{-\frac{x^2}{2}}x\,dx = \sqrt{2}\int_0^\infty e^{-\frac{x^2}{2}}\,dx$$
$$= \sqrt{2}\frac{1}{2}\sqrt{2\pi}\underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx}_{=1} = \sqrt{\pi}$$

so

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

Typical examples where Gamma random variables are used to model non-negative quantities such as for example time until death or time between successive insurance claims).
We can use the two parameters α and β to adjust the mean and variance



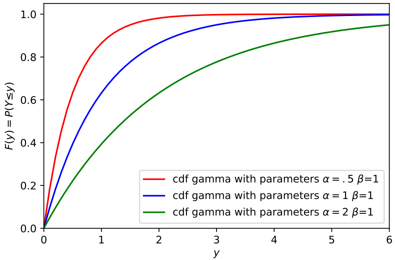


Figure: The gamma RV with Left:PDF and Right:CDF

Mean and Variance of Gamma RV

Moments of Gamma RV

For a gamma random variable Y with parameters α and β we have

$$E[Y] = \alpha\beta \qquad V(Y) = \alpha\beta^2$$

Proof

$$E[Y] = \int_0^\infty y \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^\alpha\Gamma(\alpha)}dy \underbrace{=}_{t=y/\beta} \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty t^\alpha\beta^\alpha e^{-t}\beta dt$$

$$= \frac{\beta}{\Gamma(\alpha)}\int_0^\infty t^\alpha e^{-t}dt = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta\alpha$$

$$E[Y^2] = \int_0^\infty y^2 \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^\alpha\Gamma(\alpha)}dy \underbrace{=}_{t=y/\beta} \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha\Gamma(\alpha)} = \beta^2\alpha(\alpha+1)$$

Examples

- The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Find the probability that an earthquake striking this region will
  - exceed 3.0 on the Richter scale
  - fall between 2.0 and 3.0 on the Richter scale.
- Suppose Y is exponential and $P(Y > 2) = 0.0821$.
  - What is the mean of Y?
  - What is $P(Y \leq 1.7)$?
- Explosive devices used in mining operations produce nearly circular craters when detonated. The radii of these craters are exponentially distributed with mean 10s feet. Find the mean and variance of the areas produced by these explosive devices.

χ^2-random variable

χ^2-random variable

A random variable Y is called a χ^2 random variable with k degrees of freedom and we write $Y \sim \chi^2(k)$ if it has the pdf

$$f(y) = \frac{y^{\frac{k}{2}-1} e^{-y/2}}{2^{k/2}\Gamma(k/2)}.$$

- This is a special case of gamma RV with $\beta = 2$ and $\alpha = k/2$ (half-integers).
- There is natural relation between χ^2-random variable and normal random variable (see later). For example if $Z \sim N(0,1)$ then $Z^2 \sim \chi^2(1)$

Examples

- Suppose Y has density $f(x) = Cx^3 e^{-x/2}$. Is Y a χ^2 random variable? Find the normalization constant C.
- The time it takes to go through a cash regsiter at a store is to be modeled by a Gamma RV. After observing a store you arrive with an empirical estimate of a waiting time of 10 minutes and a standard deviation of of 8 minutes. How should you pick the paramter α and β
- Let X be a Gamma random variable with parameters $\alpha = 2$ and $\beta = 3$ (rate parameter), i.e.
  1. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.
  2. Compute the probability $P(X > 2)$.