STAT 315: Uniform Random Variables

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Uniform Random Variables

A continuous random variable U is uniform if

- U takes values in some interval $[a, b]$.
- The probability that U takes value in some sub-interval $[c, d]$ (contained in $[a, b]$) only depends on the length of the interval $d - c$.

The PDF is constant on the interval $[a, b]$

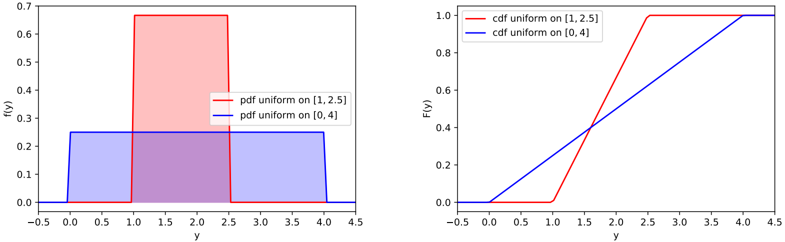

Figure: The PDF and CDF of 2 uniform RV on $[0, 4]$ and $[1, 2.5]$

PDF and CDF for uniform RV

PDF and CDF for the Uniform RV

If U is uniform on $[a, b]$ then the PDF is constant on $[a, b]$:

$$f(y) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq y \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$P(c \leq Y \leq d) = \int_c^d \frac{1}{b-a} dy = \frac{d-c}{b-a} \quad \text{if } [c, d] \subset [a, b]$$

The CDF is given by

$$F(y) = \begin{cases} 0 & \text{if } y \leq a \\ \int_a^y \frac{1}{b-a} dy = \frac{y-a}{b-a} & \text{if } a \leq y \geq b \\ 1 & \text{if } y \leq b \end{cases}$$

Notation: Write $X \sim U([a, b])$. Often use the letter U if $U \sim U([0, 1])$

Mean and Variance of uniform RV

Mean and Variance

If U is uniform on the interval $[a, b]$

$$E[U] = \frac{a+b}{2} \qquad V[U] = \frac{(b-a)^2}{12}$$

$$E[U] = \int_a^b \frac{x}{b-a} dx = \frac{1}{2}\frac{(b^2-a^2)}{b-a} = \frac{1}{2}\frac{(b-a)(b+a)}{b-a} = \frac{1}{2}(a+b)$$

$$E[U^2] = \int_a^b \frac{x^2}{b-a} dx = \frac{1}{3}\frac{(b^3-a^3)}{b-a} = \frac{1}{3}\frac{(b-a)(a^2+ab+b^2)}{b-a}$$
$$= \frac{1}{3}(a^2+ab+b^2)$$

$$V(U) = \frac{1}{3}(a^2+ab+b^2) - \frac{1}{4}(a^2+2ab+b^2) = \frac{1}{12}(a^2-2ab+b^2)$$

Practice Problems: Continuous Uniform Distribution

1. Let $Y \sim \mathrm{U}([1, 10])$.
   1. Write the probability density function $f_Y(y)$.
   2. Derive formulas for $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

2. Let $X \sim \mathrm{U}([0, 1])$.
   1. Find $\mathbb{P}(X > 0.7)$.
   2. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

3. A bus arrives uniformly at random between 8:00 and 8:30 AM.
   1. What is the probability that it arrives after 8:20 AM?
   2. What is the expected arrival time?

4. Suppose $Z \sim \mathrm{U}([2, 5])$. Find $\mathbb{P}(Z < 3.5)$ and $\mathbb{P}(1.5 < Z < 4.5)$.

Random number generator

Any computer system contains a (pseudo-random) number generator which is an algorithm (hard one!) which generates observed values for a uniform random variable $U \sim U([0,1])$. Usually the command is "rand"

```
# prompt: random number generator

import numpy as np
import random

np.random.rand(10)
```

```
array([0.32006209, 0.41149266, 0.6831682 , 0.2066693 , 0.22082369,
       0.75809193, 0.59138499, 0.55586962, 0.49877951, 0.76689334])
```

This is the only source of randomness on your computer and all random variable simulated on computer are derived from "rand". (More on this in Chapter 6)

Example: To flip a fair coin on a computer do the following:
if $U \leq \frac{1}{2}$ return 'Tails'
else return 'Heads'.