STAT 315: Random Variables and Expectation

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What is a random variable?

- $S = \{a_1, a_2, \cdots\}$ is the sample space, i.e. the list of all possible outcomes of a (random) experiment.
- $A \subset S$ is an event which you can think as an "observation": does the random experiment we observe belong to A.
- **A random variable Y is a measurement on the random experiment. This means that to each outcome a_i you assign a real number.**
  Usually we use capital letters X, Y_1, Y_2, and so on to denote random variables.

Example: Roll a pair of dice. $S = \{(i,j)\}$ with $1 \le i,j \le 6$

1. $X =$ the sum of the two dice. X takes values between 2 and 12
2. $Y =$ the number of odd numbers on the dice. Y takes values 0, 1, 2
3. $Z=$ the number on the first dice times the square of the number of the second dice. E.g $(3,6)$ gives $Z = 3 \times 36 = 108$
4. $\cdots$

How to describe a random variable?

The probability distribution function (= pdf) of a random variable

For a random variable Y taking the value y the probability distribution of a random variable is

$$p(y) \equiv P(Y = y) = \sum_{a_i : Y = y} P(a_i) \tag{1}$$

the sum of the probabilities of the sample points that assign the value y.

Example: If you toss three coins and $X=$ the number of HEADS then

$$P(X = 0) = P(\{[T, T, T]\}) = \frac{1}{8}$$

$$P(X = 1) = P(\{[H, T, T], [T, H, T], [T, T, H]\}) = \frac{3}{8}$$

$$P(X = 2) = P(\{[T, H, H], [H, T, H], [H, H, T]\}) = \frac{3}{8}$$

$$P(X = 3) = P(\{[H, H, H]\}) = \frac{1}{8}$$

Properties and graphical representation of the pdf

Properties of the pdf

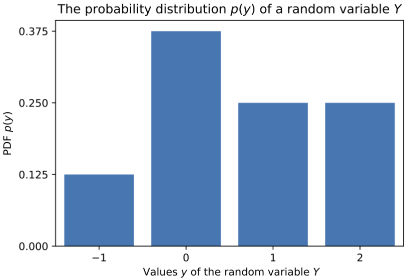
For any random variable Y we must have

1. $0 \leq p(y) \leq 1$ (positivity).
2. $\sum_y p(y) = 1$ (normalization).

Table

y	$p(y)$
-1	$1/8$
0	$3/8$
1	$1/4$
2	$1/4$
	sum $=1$

Histogram



Expected value of a random variable

The expected value (or mean) $E[Y]$ of a random variable Y is the average of the values that the random variable takes.

Expected value of Y

The expected value of Y is given by

$$E[Y] = \sum y\, p(y) = \sum_y y\, P(Y = y)\,.$$

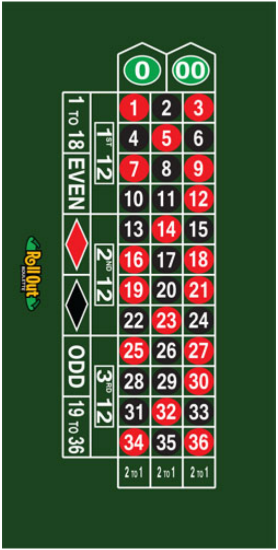
Example: If Y takes values $-1, 0, 1, 2$ with pdf

$$P(Y = -1) = \frac{1}{8}, P(Y = 0) = \frac{3}{8}, P(Y = 1) = \frac{1}{4}, P(Y = 2) = \frac{1}{4}$$

then

$$E[Y] = (-1) \times \frac{1}{8} + 0 \times \frac{3}{8} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} = \frac{5}{8}$$

Bets at American Roulette



Bets	Payout
1 number	35 to 1
2 numbers	17 to 1
4 numbers	8 to 1
black/red	1 to 1
odd/even	1 to 1
columns	2 to 1
group of 12	2 to 1

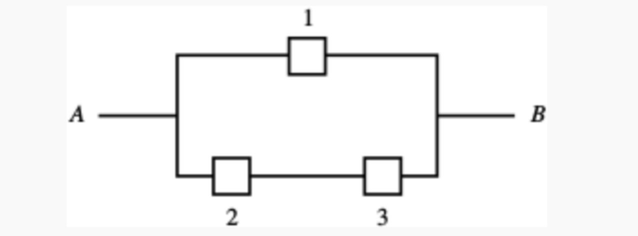
General rule: For a bet on a group of n numbers the payout is

$$\frac{36}{n} - 1$$

Which one should you bet on?

Examples

- What is the average number of H when you flip 3 coins.
- Among a group of 3 men and three women you select a group of 2. Let Y be the number of women in the group. Find the probability distribution of Y and $E[Y]$.
- Let X be the number of open paths from A to B in the following circuit (each gate open with probability .6). Find the probability distribution of X and $E[X]$.



The indicator random variable

- For an event $A \subset S$ one can always write the probability $P(A)$ has an expected value.
- Define the indicator random variable X_A as

$$X_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occurs} \end{cases}$$

- The PDF of X_A is

$$P(X=1) = P(A)\,, \quad P(X=0) = 1 - P(A)$$

- The expected value of X_A is

$$E[X_A] = 0 \times P(X=0) + 1 \times P(X=1) = P(X=1) = P(A)$$

Probabilities are expectations: $E[X_A] = P(A)$

Classification task in CS: Character recognition

https://en.wikipedia.org/wiki/MNIST_database

MNIST: Data base of 60'000 handwritten digits.

In supervised learning tasks one build algorithms which should recognize a digit from the picture (28×28 pixels, each one on a gray scale from 1 to 9).

- Sample space $= \{$images of digits$\}$
- Each algorithm assigns to each image a digit.
- Success rate RV

$$X = \begin{cases} 1 & \text{if correct} \\ 0 & \text{if not} \end{cases}$$

$$E[X] = P(\text{algorithm is correct})$$