Appendix - For Online Publication Only

This appendix provides missing proofs, details for Section 4, and comparisons with existing decomposition results.

B. Other proofs

Proof of Proposition 3.1. We show (ii)((i) follows similarly). Let i and s_i be fixed. Then from the discussion before the proposition, $w^{(i)}$ is concave in s_i for all i. Thus there exists a Nash equilibrium. We next show that $\Phi(s) = \sum_i \max_{s_i \in S_i} w^{(i)}(s_i, s_{-i})$ is strictly convex. Let t', $t'' \in S$ be given. Then u', $u'' \in S$ be given such that $w^{(i)}(u'_i, t'_{-i}) = \max_{s_i \in S_i} w^{(i)}(s_i, t'_{-i})$ and $w^{(i)}(u''_i, t''_{-i}) = \max_{s_i \in S_i} w^{(i)}(s_i, t''_{-i})$ for all i. Let $\alpha \in (0,1)$ and t^* be such that $w^{(i)}(t^*_i, ((1-\alpha)t' + \alpha t'')_{-i}) = \max_{s_i \in S_i} w^{(i)}(s_i, ((1-\alpha)t' + \alpha t'')_{-i})$ for all i. Then we have

$$
\begin{aligned}
&(1-\alpha)\Phi(t') + \alpha\Phi(t'') = (1-\alpha)\sum_i w^{(i)}(u'_i, t'_{-i}) + \alpha \sum_i w^{(i)}(u''_i, t''_{-i}) \\
&\geq (1-\alpha)\sum_i w^{(i)}(t^*_i, t'_{-i}) + \alpha \sum_i w^{(i)}(t^*_i, t''_{-i}) > \sum_i w^{(i)}(t^*_i, (1-\alpha)t'_{-i} + \alpha t''_{-i}) \\
&= \Phi((1-\alpha)t' + \alpha t'').
\end{aligned}
$$

Thus $\Phi_f(s)$ is strictly convex and the minimizer of Φ_f is unique. Since the Nash equilibrium is a minimizer of Φ_f, the Nash equilibrium is unique.

□

Proof of Corollary 3.1. Let $f = w + h$, where h is a non-strategic game. Then, $w^{(1)}(\sigma_1, \sigma_2)$ is convex in σ_2 and $w^{(2)}(\sigma_1, \sigma_2)$ is convex in σ_1. By Proposition 3.1, the set of Nash equilibria is convex. Suppose that f has two distinct Nash equilibria, ρ^* and σ^*, where $\rho^* \neq \sigma^*$. Then, for all $t \in (0,1)$, $(1-t)\rho^* + t\sigma^*$ is a Nash equilibrium since the set of Nash equilibria is convex. This contradicts Condition **(N)** because of Lemma 2.2 in Quint and Shubik (1997). □

Proof of Proposition 3.2. We let

$$\mathcal{D} := \{f \in \mathcal{L} : f^{(i)}(s) := \sum_{l \neq i} \zeta_l(s_{-l}) \text{ for all } i\}$$

We first show that

$$\mathcal{B} = (\mathcal{Z} + \mathcal{E}) \cap (\mathcal{I} + \mathcal{E}) = \mathbf{S}(\mathbf{P}(\mathcal{L})) + \mathcal{E}.$$

Let $f \in (\mathcal{Z}+\mathcal{E}) \cap (\mathcal{I}+\mathcal{E})$. Then, $f = g_1 + h_1$, for $g_1 \in \mathcal{Z}$ and $h_1 \in \mathcal{E}$, and $f = g_2 + h_2$, for $g_2 \in \mathcal{I}$ and $h_2 \in \mathcal{E}$. Thus, we have

$$g_1 + h_1 = g_2 + h_2, \tag{1}$$

and applying $\mathcal{S}$ to (1), we obtain

$$f = \mathcal{S}(h_1 - h_2) + h_2.$$

Thus, since $h_1 - h_2 \in \mathbf{P}(\mathcal{L})$, $f \in \mathbf{S}(\mathbf{P}(\mathcal{L})) + \mathcal{E}$. Conversely, let $f \in \mathbf{S}(\mathbf{P}(\mathcal{L})) + \mathcal{E}$. Obviously, $f \in \mathcal{I} + \mathcal{E}$. In addition, $f = \mathbf{S}(\mathbf{P}(g)) + h_1$, for $g \in \mathcal{L}$ and $h_1 \in \mathcal{E}$. Thus,

$$f = \mathbf{S}(\mathbf{P}(g)) + h_1 = -(\mathbf{I} - \mathbf{S})(\mathbf{P}(g)) + \mathbf{P}(g) + h_1 \in \mathcal{Z} + \mathcal{E}.$$

This shows that

$$(\mathcal{Z} + \mathcal{E}) \cap (\mathcal{I} + \mathcal{E}) = \mathbf{S}(\mathbf{P}(\mathcal{L})) + \mathcal{E}.$$

Note that

$$\begin{aligned}
\mathbf{S}(\mathbf{P}(\mathcal{L})) + \mathcal{E} &= \{f : f^{(i)} = \sum_{l=1}^{n} \zeta_l(s_{-l}) \text{ for some } \{\zeta_l\}_{l=1}^{n} \text{ and for all } i\} + \mathcal{E} \\
&= \{f : f^{(i)} = \sum_{l \neq i} \zeta_l(s_{-l}) \text{ for some } \{\zeta_l\}_{l=1}^{n} \text{ and for all } i\} + \mathcal{E} \\
&= \mathcal{D} + \mathcal{E}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
&(\sum_{l \neq 1} \zeta_l(s_{-l}), \sum_{l \neq 2} \zeta_l(s_{-l}), \cdots, \sum_{l \neq n} \zeta_l(s_{-l})) \\
&\sim (\sum_{l=1}^{n} \zeta_l(s_{-l}), \sum_{l=2}^{n} \zeta_l(s_{-l}), \cdots, \sum_{l=1}^{n} \zeta_l(s_{-l})).
\end{aligned}$$

Hence, the first result follows from $\mathcal{D} + \mathcal{E} = (\mathcal{I} + \mathcal{E}) \cap (\mathcal{Z} + \mathcal{E})$. For the second result, observe that

$$
\begin{aligned}
(\sum_{l\neq 1}\zeta_l, \sum_{l\neq 2}\zeta_l, \cdots, \sum_{l\neq n}\zeta_l) &\sim (\sum_{l\neq 1}\zeta_l-(n-1)\zeta_1, \sum_{l\neq 2}\zeta_l-(n-1)\zeta_2, \cdots, \sum_{l\neq n}\zeta_l-(n-1)\zeta_n)\\
&= (\sum_{l\neq 1}(\zeta_l-\zeta_1), \sum_{l\neq 2}(\zeta_l-\zeta_2), \cdots, \sum_{l\neq n}(\zeta_l-\zeta_n))\\
&= (\sum_{l>1}(\zeta_l-\zeta_1), \zeta_1-\zeta_2, \zeta_1-\zeta_3, \cdots, \zeta_1-\zeta_n)\\
&+ (0, \sum_{l>2}(\zeta_l-\zeta_2), \zeta_2-\zeta_3, \cdots, \zeta_2-\zeta_n) + \cdots\\
&+ (0, 0, \cdots, \sum_{l>n-1}(\zeta_l-\zeta_{n-1}), \zeta_{n-1}-\zeta_n)\\
&= \sum_{i=1}^{n}\sum_{l>i}^{n}(0, \cdots, 0, \underbrace{-\zeta_i+\zeta_j}_{i-\text{th}}, 0, \cdots 0, \underbrace{\zeta_i-\zeta_j}_{j-\text{th}}, 0, \cdots, 0)\\
&= \sum_{i<j}(0, \cdots, 0, \underbrace{-\zeta_i+\zeta_j}_{i-\text{th}}, 0, \cdots 0, \underbrace{\zeta_i-\zeta_j}_{j-\text{th}}, 0, \cdots, 0).
\end{aligned}
$$

□

Proof of Corollary 3.2. (i) This immediately follows from Proposition 3.2. (ii) From the second part of Corollary 3.2, $(s_1^*, s_2^*) \in (\arg\max_{s_1}\zeta_2(s_1), \arg\max_{s_2}\zeta_1(s_2))$ is a Nash equilibrium. If there are two distinctive maximizers, then since the set of maximizers is convex, there exist infinitely many Nash equilibria, contradicting Condition **(N)** again by Lemma 2.2 in Quint and Shubik (1997). Thus, the maximizer is unique and constitutes the strictly dominant Nash equilibrium. □

Proof of Proposition 3.3. Let $d\sigma_i(s_i) = \frac{1}{m(S_i)}dm_i(s_i)$ be player i' uniform mixed strategy. We define a uniform mixed strategy profile as a product measure of uniform mixed strategies: i.e.,

$$d\sigma(s) = \prod_i d\sigma_i(s_i).$$

Let i and s_i be fixed. We show that

$$f^{(i)}(s_i, \sigma_{-i}) = 0.$$

	Identity payoff Normalized $\mathcal{I} \cap \mathcal{N}$	Zero-sum Normalized $\mathcal{Z} \cap \mathcal{N}$	Both Potential and Zero-sum $\mathcal{B}$
Dimensions	$\frac{(l-1)l}{2}$	$\frac{(l-2)(l-1)}{2}$	$2l-1$
Basis Games	$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Table 1: **Dimensions of subspaces and basis games for two-player symmetric games**

Then, the desired result follows since $f^{(i)}(s_i, \sigma_{-i}) = 0 = f^{(i)}(\sigma_i, \sigma_{-i})$ for all i and s_i; hence, $f^{(i)}(\sigma_i, \sigma_{-i}) = \max_{s_i} f^{(i)}(s_i, \sigma_{-i})$ for all i. First, by the definition of the mixed strategy extension,

$$f^{(i)}(s_i, \sigma_{-i}) = \int_{s_{-i} \in S_{-i}} f^{(i)}(s_i, s_{-i}) \prod_{l \neq i} d\sigma_l(s_l).$$

If f is a zero-sum normalized game, then

$$f^{(i)}(s_i, \sigma_{-i}) = -\int_{s_{-i} \in S_{-i}} \sum_{j \neq i} f^{(j)}(s_j, s_{-j}) \prod_{l \neq i} d\sigma_l(s_l) = -\sum_{j \neq i} \int_{s_{-i} \in S_{-i}} f^{(j)}(s_j, s_{-j}) \prod_{l \neq i} d\sigma_l(s_l) = 0$$

where the last equality follows from the normalization, $\int_{s_l \in S_l} f^{(l)}(s_l, s_{-l}) d\sigma_l(s_l) = 0$ for all l and Fubini's Theorem. If f is an identical interest game, then similarly

$$f^{(i)}(s_i, \sigma_{-i}) = \int_{s_{-i} \in S_{-i}} v(s_i, s_{-i}) \prod_{l \neq i} d\sigma_l(s_l) = 0$$

where the last equality again follows from the normalization, $\int_{s_l \in S_l} v(s_l, s_{-l}) d\sigma_l(s_l) = 0$ for all l. Thus, we obtain the desired result. $\square$

C. Details for Section 4

C.1. Finite strategy games

Lemma C.1. *We have the following results:*
(i) The set of games $\{S^{(ij)}\}_{i=1,\cdots,l,j>i}$ *forms a basis set for* $\mathcal{I} \cap \mathcal{N}$
(ii) The set of games $\{Z^{(ij)}\}_{i=2,\cdots,l,j>i}$ *forms a basis set for* $\mathcal{I} \cap \mathcal{N}$
(iii) The set of games $\{D^{(i)}\}_{i=1,\cdots,l-1}, \{E^{(i)}\}_{i=1,\cdots,l}$ *forms a basis set for* $\mathcal{B}$

Proof. (i) We note that there are precisely $\frac{l(l-1)}{2}$ number of different $S^{(ij)}$'s. Thus, we only need to show that these $S^{(ij)}$'s are independent. Let S be

$$S := \sum_{i=1}^{l} \sum_{j=i+1}^{l} \alpha^{(ij)} S^{(ij)}.$$

Then it is easy to check that $S_{ij} = \alpha^{(ij)}$.Thus if $S = \mathbf{O}$, then $\alpha^{(ij)} = 0$ for all i, j.
(ii) Again we note that there are precisely $\frac{(l-2)(l-1)}{2}$ number of different $Z^{(ij)}$'s. Let Z be

$$Z := \sum_{i=2}^{l} \sum_{j=i+1}^{l} \zeta^{(ij)} Z^{(ij)}.$$

Then it is also easy to check that $Z_{ij} = -\zeta^{(ij)}$.Thus if $Z = \mathbf{O}$, then $\zeta^{(ij)} = 0$ for all i, j.
(iii) Again we note that there are precisely $l - 1$ number of different $D^{(i)}$'s and l number of different $E^{(i)}$'s. Let K be

$$K = \sum_{i=1}^{l-1} \delta_i D^{(i)} + \sum_{i=1}^{l} \eta_i E^{(i)}$$

Then if $K = \mathbf{O}$, then $\eta_i = 0$ for all i (because the last row of K is given by $(\eta_1, \cdots, \eta_l)$) and this, in turn, implies that $\delta_i = 0$ for all i. □

We would have the following results.

Proposition C.1. *We have the following results:*
(i) Suppose that $\gamma_{ij} < 0$ for all i, j. Then $\#(G) = 1$.
(ii) Suppose that $\gamma_{ij} > 0$ for all i, j. Suppose that $\delta_i \geq 0$ for all i and $\underline{\gamma} > \bar{\delta} + \bar{\zeta}$. Then $\#(G) = 2^l - 1$.

Proof of Proposition C.1 (ii). We will show that G satisfies the total band wagon property defined by Kandori and Rob (1998). Then for any $A \subset \{1, 2, \cdots, l\}$, there exists a unique Nash equilibrium, which is completely mixed in A and thus there exist precisely $2^l - 1$ Nash equilibria (See Kandori and Rob (1998)). Thus, we will show that for all $q \in \Delta$, $BR(q) \subset \Sigma_q$, where $BR(q)$ is the set of all pure strategy best responses for q. Suppose that there exists $q \in \Delta$ such that $BR(q) \not\subset \Sigma_q$. Then we must have $\Sigma_q \neq \{1, \cdots, l\}$ (the set of all pure strategies) and there exists $k \notin \Sigma_q$

such that

$$e_k \cdot Gq \geq q \cdot Gq$$

We define $\gamma_{ji} = \gamma_{ij}$ for $j > i$. First observe that we have

$$q \cdot Sq = \sum_{i<j} \gamma_{ij}(q_i - q_j)^2, \quad e_k \cdot Sq = \sum_{j<k} \gamma_{jk}(q_k - q_j) + \sum_{j>k} \gamma_{kj}(q_k - q_j) = \sum_{j \neq k} \gamma_{kj}(q_k - q_j)$$

Next we define $\zeta_{ji} = -\zeta_{ij}$ for $j > i$. Again observe that we have

$$q \cdot Zq = 0, \quad e_1 \cdot Sq = \sum_{i<j} \zeta_{ij}(q_j - q_i), \quad e_k \cdot Sq = \sum_{j \neq k} \zeta_{ij}(q_1 - q_j)$$

Thus

$$\begin{aligned} e_1 \cdot Sq &\leq \sum_{i<j} \max_{i<j} |\zeta_{ij}||q_j - q_i| \leq \max_{i<j} |\zeta_{ij}| \sum_{i<j} \leq \bar{\zeta} \\ e_k \cdot Sq &\leq \sum_{j \neq k} \max_{i<j} |\zeta_{ij}||q_1 - q_j| \leq \max_{i<j} |\zeta_{ij}| \sum_{i<j} \leq \bar{\zeta} \end{aligned}$$

Next we let $d = (\delta_1, \cdots, \delta_l)^T$ and find that

$$q \cdot Dq = q \cdot d \geq 0, \quad e_k \cdot Dq = \delta_k$$

since $\delta_i \geq 0$ for all i. Then since $k \notin \Sigma_q$ so $q_k = 0$. Thus

$$\begin{aligned} &0 \leq q \cdot Sq + q \cdot Zq + q \cdot Dq = q \cdot Gq \leq e_k \cdot Gq = \sum_{j \neq k} \gamma_{kj}(q_k - q_j) + \bar{\zeta} + \bar{\delta} \\ &= -\sum_{j \neq k} \gamma_{kj} q_j + \bar{\zeta} + \bar{\delta} \leq -\sum_{j \in \Sigma_q} \gamma_{kj} q_j + \bar{\zeta} + \bar{\delta} \leq -\underline{\gamma} + \bar{\zeta} + \bar{\delta} < 0 \end{aligned}$$

which is a contradiction. The last inequality in the above follows from

$$\sum_{j \in \sum_q} \gamma_{kj} q_j \geq \sum_{j \in \sum_q} \min_{j \in \sum_q} \gamma_{kj} q_j \geq \sum_{j \in \sum_q} \min_{j \neq k} \gamma_{kj} q_j \geq \min_{j \neq k} \gamma_{kj} \sum_{j \in \sum_q} q_j = \min_{j \neq k} \gamma_{kj} \geq \underline{\gamma}$$

□

To show (i) of Proposition C.1, we recall the following definitions (from Hofbauer and Sandholm

(2009)).

Definition C.1. We say that
(i) a symmetric game G is stable if $(q-p)\cdot G(q-p) \leq 0$ for all $p, q \in \Delta$
(ii) a symmetric game G is strictly stable if $(q-p)\cdot G(q-p) < 0$ for all $p \neq q \in \Delta$
(iii) a symmetric game G is null-stable if $(q-p)\cdot G(q-p) = 0$ for all $p, q \in \Delta$

Next we have the following well-known observation.

Lemma C.2. *If p satisfies*

$$(q-p)\cdot Gq < 0 \textit{ for all } q \neq p \in \Delta$$

then p is a unique Nash equilibrium for a symmetric game, G.

Proof. Since G is finite, there exist a Nash equilibrium, say p'. We will show that $p' = p$. Suppose that $p' \neq p$. Then we find

$$p \cdot Gp' > p' \cdot Gp'$$

which shows that p' is not a Nash equilibrium, a contradiction. Thus we must have $p' = p$. And this also shows that there cannot exist any other Nash equilibrium. $\square$

We have the following characterization for the strict stability of G.

Lemma C.3. *Suppose that G is given by (31).*
(i) G is strictly stable if $\gamma_{ij} < 0$ for all $i < j$.
(iii) G is null stable if $\gamma_{ij} = 0$ for all $i < j$.

Proof. (i) Let $T\Delta$ be the tangent space of Δ and $z \neq 0$ and $z \in T\Delta$. Then since $G = S + Z + B$, $Bz = \mathbf{0}$,and $z \cdot Zz = 0$, we have

$$z \cdot Gz = z \cdot Sz = z \cdot \sum_{i<j} \gamma_{ij} S^{(ij)} z = \sum_{i<j} z \cdot S^{(ij)} z = \sum_{i<j} \gamma_{ij}(z_i - z_j)^2 \leq 0$$

because $\gamma_{ij} < 0$ for all $i < j$. If $\sum_{i<j} \gamma_{ij}(z_i - z_j)^2 = 0$, $\gamma_{ij}(z_i - z_j)^2 = 0$ for all $i < j$ and thus $z_i - z_j = 0$ for all $i > j$ which is a contradiction to $z \neq 0$. Thus we have $z \cdot Gz < 0$. (ii) Let $z \in T\Delta$. If $\gamma_{ij} = 0$, then we again have

$$z \cdot Gz = \sum_{i<j} \gamma_{ij}(z_i - z_j)^2 = 0$$

□

Proof of Proposition C.1 (i). Suppose that $\gamma_{ij} < 0$ for all $i > j$. Since G is a finite game, there exists a NE, p^*, for G. Since G is strictly stable, for all $q \neq p^*$, we have

$$(q - p^*) \cdot G(q - p^*) < 0 \text{ for all } q \neq p^* \in \Delta$$

Thus

$$(q - p^*) \cdot Gq < (q - p^*) \cdot Gp^* \leq 0$$

where the last inequality follows from p^* is a NE. Thus we find that $\#(G) = 1$. □

C.2. Contest games

First note that $s = (0, 0, \cdot, 0)$ cannot be a Nash equilibrium since any player i can deviate to $s_i > 0$. Thus we let

$$S = \{(s_1, \cdots, s_n) : s_i \geq 0 \text{ for all } i, \text{ and } s_j > 0 \text{ for some } j\}.$$

Lemma C.4. *Let i be fixed and $s_i \geq 0$. Then $w^{(i)}(s_i, \cdot) : S_{-i} \to \mathbb{R}$ is convex.*

Proof. We will show that $p^{(i)}(s_i, \cdot) : S_{-i} \to \mathbb{R}$ is convex and then the desired result follows. Suppose that $s_i > 0$. Define $g : s_{-i} \mapsto \sum_{l \neq i} s_l$ and $h : t \mapsto \frac{s_1}{s_1 + t}$. Then g is convex, h is convex and decreasing, thus $p^{(i)}(s_i, \cdot)$ is convex. If $s_i = 0$, then

$$p^{(i)}(0, s_{-i}) = \begin{cases} \frac{1}{n}, & \text{if } s_{-i} = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Thus $p^{(i)}(0, \cdot)$ is convex for all $s_{-i} \neq 0$. Thus we obtain the desired result. □

Since it is known that the rent-seeking game admits a Nash equilibrium, Proposition 3.1 and Lemma C.4 show that the set of Nash equilibrium for the rent-seeking game is convex. Let $b_i^\circ(s_{-i})$ be the best response when an interior solution occurs. That is, $b_i^\circ(s_{-i})$ satisfies

$$c_i(b_i^\circ(s_{-i}) + \sum_{l \neq i} s_l)^2 = \sum_{l \neq i} s_l.$$

Then

$$\Phi_f(s) = \sum_{l=i} w^{(i)}(\max\{b_i^\circ(s_{-i}), 0\}, s_{-i})$$

For $P \subset \{1, \cdots, n\}$ such that $|P| \geq 2$, we define

$$w_P^{(i)}(s_i, s_{-i}) = (p^{(i)}(s_i, s_{-i}) - \frac{1}{|P|}) - \frac{1}{|P|-1} \sum_{j \neq i, j \in P} (c_i s_i - c_j s_j)$$

for $s \in \prod_{i=1} S_i$.

Lemma C.5. *Suppose that s^* is a Nash equilibrium and $s_i^* > 0$ for all $i \in P$ and $s_i^* = 0$ for all $i \notin P$ where $P \subset \{1, \cdots, n\}$. Let $s_P^* = (s_i^*)_{i \in P}$. Then we have*

$$\Phi_f(s^*) = \sum_{i \in P} w_P^{(i)}(b_i^\circ(s_{P,-i}^*), s_{P,-i}^*)$$

Proof. Let $P \subset \{1, \cdots, n\}$ such that for all $i \in P$, $s_i > 0, b_i(s_{-i}) > 0$ and for all $i \notin P$, $s_i = 0, b_i(s_{-i}) = 0$. Then we have

$$\begin{aligned}
\sum_{i=1}^{n} w^{(i)}(b_i(s_{-i}), s_{-i}) =& \sum_{i \in P} w^{(i)}(b_i^0(s_{-i}), s_{-i}) + \sum_{i \notin P} w^{(i)}(0, s_{-i}) \\
=& \sum_{i \in P} [p^{(i)}(b_i^\circ(s_{-i}), s_{-i}) - c_i b_i^\circ(s_{-i})) + \frac{1}{n-1} \sum_{j \neq i, j \in P} c_j s_j] \\
&+ \sum_{i \notin P} \frac{1}{n-1} \sum_{j \neq i, j \in P} c_j s_j - 1 \\
=& \sum_{i \in P} w_P^{(i)}(b_i^\circ(s_{P,-i}), s_{P,-i})
\end{aligned}$$

where we use $b_i^\circ(s_{-i}) = b_i^\circ(s_{P,-i})$, since $b_i^\circ(s_{-i})$ depends $\sum_{l \neq i} s_l$. Using this, we obtain the desired result. $\square$

Lemma C.5 leads us to define

$$\begin{aligned}\Phi_P^\circ(s) :=& \sum_{i\in P} w_P^{(i)}(b_i^\circ(s_{-i}), s_{-i}) \\ =& \frac{1}{|P|-1}\sum_{i\in P} c_i(\sum_{i\in P}\sum_{l\neq i} s_l) - 2\sum_{i\in P}\sqrt{c_i}\sqrt{\sum_{l\neq i} s_l} + |P| - 1 \end{aligned} \tag{2}$$

for $s \in S(P)$.

Lemma C.6. $\Phi_P^\circ(s) : S(P) \to \mathbb{R}$ *is strictly convex.*

Proof. From (2), it is enough to consider the following function:

$$\Psi(s) := \sum_{i=1}^n \alpha_i h(\sum_{l\neq i} s_l)$$

where $\alpha_i > 0$ and h is strictly convex. We will show that Ψ is strictly convex. Let $s, t \in S_+$ and $s \neq t$ and $\rho \in (0,1)$. Then for some k, $\sum_{l\neq k} s_l \neq \sum_{l\neq k} t_l$. Otherwise, if $\sum_{l\neq i} s_l = \sum_{l\neq i} t_l$, then $\sum_l s_l = \sum_l t_l$, which again implies $s_i = t_i$, a contradiction. Thus from the strict convexity of h, we have

$$h((1-\rho)\sum_{l\neq k} s_l + \rho\sum_{l\neq k} t_l) > (1-\rho)h(\sum_{l\neq k} s_l) + \rho h(\sum_{l\neq k} t_l)$$

and

$$\Psi((1-\rho)s+\rho t) = \sum_{i=1}^n \alpha_i h(\sum_{l\neq i}(1-\rho)s_l+\rho t_l) > \sum_{i=1}^n \alpha_i (1-\rho)h(\sum_{l\neq i} s_i)+\rho h(\sum_{l\neq i} t_l) = (1-\rho)\Psi(s)+\rho\Psi(t)$$

□

Lemma C.7. *Suppose that s^* and t^* are Nash equilibria for a rent-seeking game defined in (34) such that $s_i^*, t_i^* > 0$ for all $i \in P$ and $s_i^*, t_i^* = 0$ for all $i \notin P$ for some $P \subset \{1, \cdots, n\}$. Then $s^* = t^*$.*

Proof. Suppose that s^* and t^* are Nash equilibria for $\Gamma^{(n)}$ such that $s_i^*, t_i^* > 0$ for all $i \in P$ and $s_i^*, t_i^* = 0$ for all $i \notin P$. Let $s_P^* := (s_i^*)_{i\in P}$ and $t_P^* := (t_i^*)_{i\in P}$. Then we have

$$0 = \Phi(s^*) = \Phi_P^\circ(s_P^*) \text{ and } 0 = \Phi(t^*) = \Phi_P^\circ(t_P^*).$$

Since $\Phi_P^\circ(s_P) \geq 0$ for all $s_P \in \prod_{i \in P} S_i$ (This is to be shown) and the strict convexity of $\Phi_P^\circ(s)$ implies that the minimum is unique. We have $s_P^* = t_P^*$, and thus $s^* = t^*$. ☐

Proposition C.2. *The Nash equilibrium for the rent-seeking game defined in (34) is unique.*

Proof. Suppose that s^* and t^* such that $s^* \neq t^*$ are Nash equilibria. Let $P' := \{i : s_i^* > 0\}$ and $P'' := \{i : t_i^* > 0\}$. Then from Lemma C.7, we must have $P' \neq P$. Since the set of Nash equilibria is convex by Lemma C.4, $\rho s^* + (1-\rho)t^*$ is a Nash equilibria for all $\rho \in [0, 1]$. Then for $0 < \rho < 1$, $(\rho s^* + (1-\rho)t^*)_i > 0$ if $i \in P'$ and $(\rho s^* + (1-\rho)t^*)_j > 0$ if $j \in P''$. Thus there are infinitely many Nash equilibrium for the set $P' \cup P''$, which is contradiction to Lemma C.7. ☐

D. Existing decomposition results

Our decomposition methods extend two kinds of existing results: (i) Kalai and Kalai (2010), (ii) Hwang and Rey-Bellet (2011); Candogan et al. (2011). First, Kalai and Kalai (2010) decompose normal form games with incomplete information and study the implications for Bayesian mechanism designs. Their decomposition is based on the orthogonal decomposition $\mathcal{L} = \mathcal{I} \oplus \mathcal{Z}$ in equation (11). Second, Hwang and Rey-Bellet (2011) similarly provide decomposition results based on the orthogonality between identical interest and zero-sum games and between normalized and non-strategic games, mainly focusing on finite games. Candogan et al. (2011) decompose finite strategy games into three components: a potential component, a nonstrategic component, and a harmonic component. When the numbers of strategies are the same for all players, harmonic components are the same as zero-sum normalized games, and their harmonic games, in this case, refer to games that are strategically equivalent to zero-sum normalized games. Also, their potential component is obtained by removing the non-strategic component from the potential part $(\mathcal{I} + \mathcal{E})$ of the games. Note that we can change our definition of zero-sum normalized games to their definition of harmonic games, with all the decomposition results remaining unchanged. Thus, their three-component decomposition of finite strategy games follows from Theorem 2.1, $\mathcal{L} = (\mathcal{I} + \mathcal{E}) \oplus (\mathcal{Z} \cap \mathcal{N})$ (see the proof of Corollary D.1 for more detail).

Corollary D.1. *We have the following decomposition.*

$$\mathcal{L} = \underbrace{((\mathcal{I}+\mathcal{E})\cap\mathcal{N})}_{\textit{Potential Component}} \oplus \underbrace{\mathcal{E}}_{\substack{\textit{Nonstrategic}\\ \textit{Component}}} \oplus \underbrace{(\mathcal{Z}\cap\mathcal{N})}_{\substack{\textit{Harmonic}\\ \textit{Component}}}$$

Proof. This proof follows from Theorem 2.1 by showing that $((\mathcal{I}+\mathcal{E})\cap\mathcal{N})\oplus\mathcal{E} = \mathcal{I}+\mathcal{E}$. First, observe that $(\mathcal{I}+\mathcal{E})\cap\mathcal{N} \subset \mathcal{I}+\mathcal{E}$, which implies that $((\mathcal{I}+\mathcal{E})\cap\mathcal{N})\oplus\mathcal{E} \subset \mathcal{I}+\mathcal{E}$. Now, let $f \in \mathcal{I}+\mathcal{E}$. Then, $f = g+h$, where $g \in \mathcal{I}$, $h \in \mathcal{E}$, and $g = (v, v, \cdots, v)$. Then, by applying the map, $\mathbf{P}$, we find that $f = \mathbf{P}(f) + (I-\mathbf{P})(f)$. Obviously, $\mathbf{P}(f) \in \mathcal{E}$. In addition, $(I-\mathbf{P})(f) = (I-\mathbf{P})(g) = (v - T_1 v,\, v - T_2 v, \cdots, v - T_n v) \in \mathcal{I}+\mathcal{E}$. Thus, $(I-\mathbf{P})(f) \in (\mathcal{I}+\mathcal{E})\cap\mathcal{N}$. □

Ui (2000) provides the following characterization for potential games:

$$f \text{ is a potential game if and only if } f^{(i)} = \sum_{\substack{M\subset N\\ M\ni i}} \xi_M \text{ for some } \{\xi_M\}_{M\subset N} \text{ for all } i \tag{3}$$

where ξ_M depends only on s_l, with $l \in M$. Let

$$\mathcal{D} := \{f \in \mathcal{L} : f^{(i)}(s) := \sum_{l\neq i} \zeta_l(s_{-l}) \text{ for all } i\}.$$

From our decomposition results, we have $\mathcal{D} \subset \mathcal{I}+\mathcal{E}$ and $\mathcal{E} \subset \mathcal{I}+\mathcal{D}$. In particular, the second inclusion holds because

$$\zeta_i(s_{-i}) = \sum_{l=1}^{n} \zeta_l(s_{-l}) - \sum_{l\neq i} \zeta_l(s_{-l}).$$

Thus, $\mathcal{D} \subset \mathcal{I}+\mathcal{E}$ implies that $\mathcal{I}+\mathcal{D}+\mathcal{E} \subset \mathcal{I}+\mathcal{E}$ and $\mathcal{E} \subset \mathcal{I}+\mathcal{D}$ implies $\mathcal{I}+\mathcal{D}+\mathcal{E} \subset \mathcal{I}+\mathcal{D}$. From this, we find

$$\mathcal{I}+\mathcal{D} = \mathcal{I}+\mathcal{D}+\mathcal{E} = \mathcal{I}+\mathcal{E} \tag{4}$$

Note that all games in $\mathcal{I}$ and in $\mathcal{D}$ satisfy Ui's condition in (3); hence, games in $\mathcal{I}+\mathcal{D}$ satisfy Ui's condition. Then, equalities in (4) show that the condition in (3) is a necessary condition for potential games. The sufficiency of Ui's condition is deduced by adding the non-strategic game

$$(\sum_{\substack{M\subset N\\ M\not\ni 1}}\xi_M,\sum_{\substack{M\subset N\\ M\not\ni 2}}\xi_M,\cdots,\sum_{\substack{M\subset N\\ M\not\ni n}}\xi_M)$$

to game f satisfying Ui's condition.

As explained in the main text, Sandholm (2010) decomposes n-player finite strategy games into 2^n components using an orthogonal projection. When the set of games consists of symmetric games with l strategies, the orthogonal projection is given by $\Gamma := I - \frac{1}{l}\mathbf{1}\mathbf{1}^T$, where I is the $l\times l$ identity matrix and $\mathbf{1}$ is the column vector consisting of all 1's. Using Γ, we can, for example, write a given symmetric game, A, as

$$A = \underbrace{\Gamma A\Gamma}_{=(\mathcal{I}\cap\mathcal{N})\oplus(\mathcal{Z}\cap\mathcal{N})} + \underbrace{(I-\Gamma)A\Gamma+\Gamma A(I-\Gamma)+(I-\Gamma)A(I-\Gamma)}_{=\mathcal{B}}.^{1} \tag{5}$$

Thus, our decompositions show that $\Gamma A\Gamma$ can be decomposed further into games with different properties—identical interest normalized games and zero-sum normalized games—and every game belonging to the second component in (5) is strategically equivalent to both an identical interest game and a zero-sum game. Sandholm (2010) also shows that a two-player game, (A, B), is potential if and only if $\Gamma A\Gamma = \Gamma B\Gamma$. If $P = (P^{(1)}, P^{(2)})$ is a non-strategic game, it is easy to see that $\Gamma P^{(1)} = O$ and $P^{(2)}\Gamma = O$, where O is a zero matrix. Thus, the necessity of the condition $\Gamma A\Gamma = \Gamma B\Gamma$ for potential games is obtained. Conversely, if $\Gamma A\Gamma = \Gamma B\Gamma$, then game (A, B) does not have a component belonging to $\mathcal{Z}\cap\mathcal{N}$ because $(\Gamma A\Gamma, \Gamma B\Gamma) \in (\mathcal{I}\cap\mathcal{N})\oplus(\mathcal{Z}\cap\mathcal{N})$. Thus, (A,B) is a potential game.

[1] In fact, for two player symmetric game, using Table 4 we can verify that

$$f_{\mathcal{B}} = (I-(I-T_1)(I-T_2))f^{(1)}, \quad f_{\mathcal{I}\cap\mathcal{N}} + f_{\mathcal{Z}\cap\mathcal{N}} = (I-T_1)(I-T_2)f^{(1)}.$$

References

Candogan, O., Menache, I., Ozdaglar, A., Parrilo, P. A., 2011. Flows and decompositions of games: Harmonic and potential games. Mathemtics of Operations Research 36 (3), 474–503.

Hofbauer, J., Sandholm, W., 2009. Stable games and their dynamics. Journal of Economic Theory 144, 1665–1693.

Hwang, S.-H., Rey-Bellet, L., 2011. Decomposition of two player games: Potential, zero-sum, and stable games. arXiv:1106.3552.

Kalai, A., Kalai, E., 2010. Cooperation and competition in strategic games with private information. EC '10 Proceedings of the 11th ACM conference on Electronic commerce.

Kandori, M., Rob, R., 1998. Bandwagon effects and long run technology choice. Games and Ecoomic Behavior 22, 30–60.

Quint, T., Shubik, M., 1997. A theorem on the number of Nash equilibria in a bimatrix game. Internation Journal of Game Theory 26, 353–359.

Sandholm, W., 2010. Decompositions and potentials for normal form games. Games and Economic Behavior 70, 446–456.

Ui, T., 2000. A Shapley value representation of potential games. Games and Economic Behavior 31, 121–135.